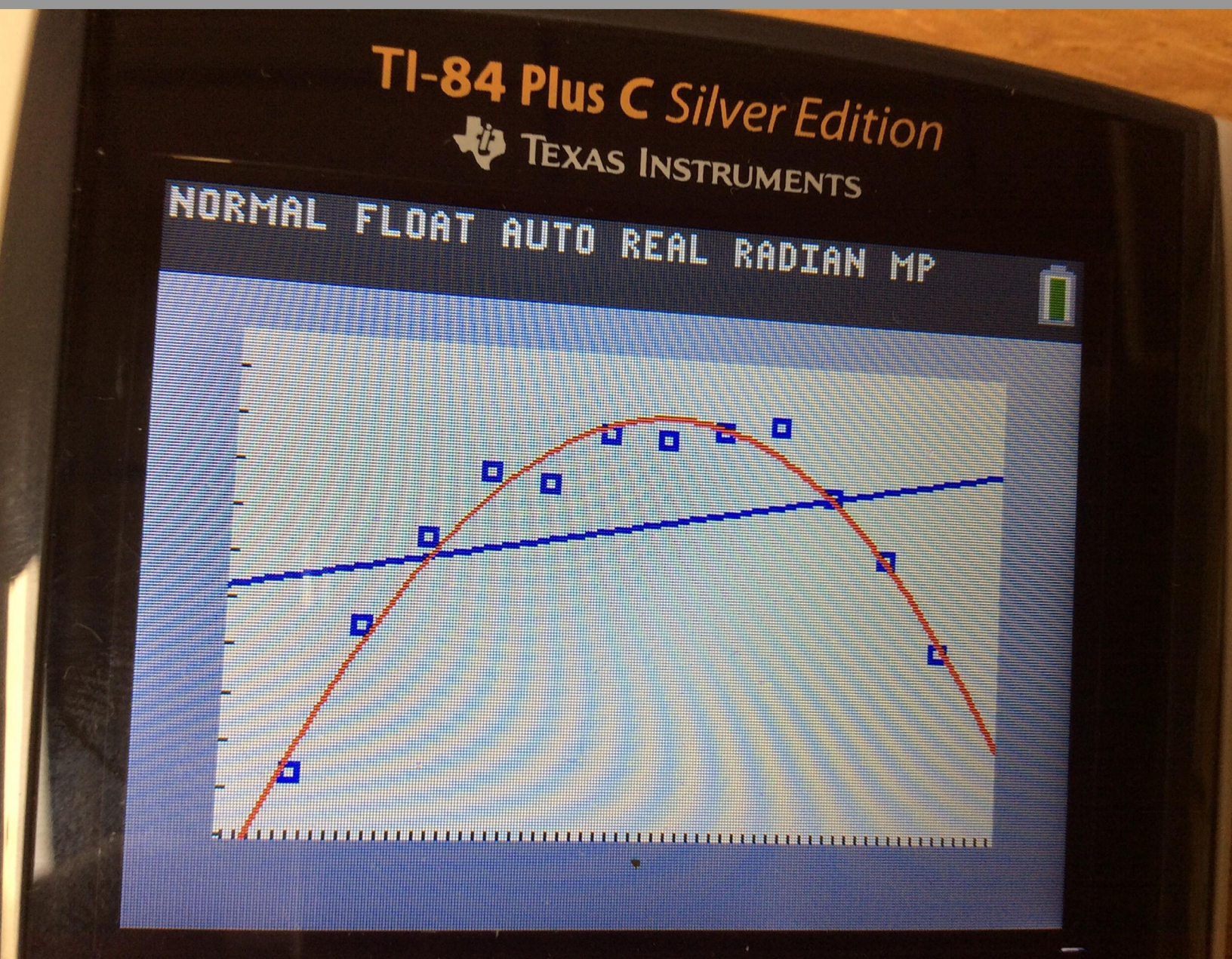


## Precalculus



# Precalculus

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John Casalinuovo

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## AUTHOR

John Casalinuovo

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## CHAPTER

## 1

# Introduction to the Graphing Calculator

## Chapter Outline

---

- 1.1 MAXIMUMS AND MINIMUMS
  - 1.2 INCREASING AND DECREASING
  - 1.3 ZEROES AND INTERCEPTS OF FUNCTIONS
  - 1.4 REFERENCES
- 

Here you will review and extend concepts about functions and graphing with the graphing calculator.

## 1.1 Maximums and Minimums

Here you will learn to identify the maximums and minimums in various graphs and be able to differentiate between global and relative extreme values.

Here are some videos to teach you the basics of the graphing calculator:

**TI 84 OPERATION VIDEOS 1 AND 2** <http://www.youtube.com/watch?v=lpH4kwamUm8> [http://www.youtube.com/watch?v=bvfU5\\_cqy8](http://www.youtube.com/watch?v=bvfU5_cqy8)

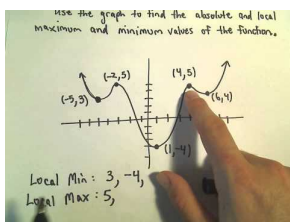
**TI 84 Operation Video 3 - Storing values** [http://tiskills.com/mod1\\_lessons/m1\\_113.php](http://tiskills.com/mod1_lessons/m1_113.php) Here are some videos detailing how graphing is done on a graphing calculator:

**TI Skills.com Graphing on the TI84 Plus Silver Edition Videos 1, 2, and 3** <http://www.youtube.com/watch?v=tIAtBsfgXxc> <http://www.youtube.com/watch?v=ChJz5mYqAjs> <http://www.youtube.com/watch?v=gH8vN1TYXio>

**TI Skills.com Graphing on the TI84 Plus Silver Edition Video 4** <http://www.youtube.com/watch?v=AXMDmOoiXZc>

When riding a roller coaster there is always one point that is the absolute highest off the ground. There are usually many other places that reach fairly high, just not as high as the first. How do you identify and distinguish between these different peaks in a precise way?

### Watch This



### MEDIA

Click image to the left or use the URL below.

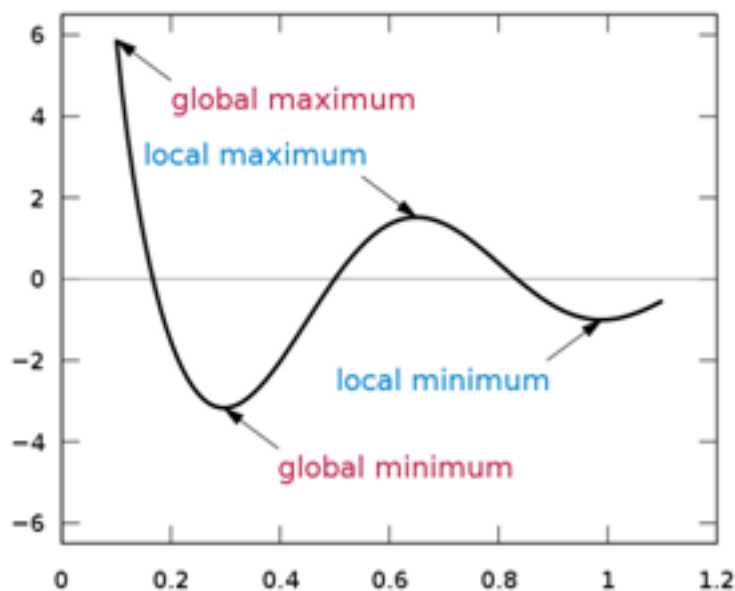
URL: <http://www.ck12.org/flx/render/embeddedobject/57938>

<http://www.youtube.com/watch?v=votVWz-wKeI>

### Guidance

A global (absolute) maximum refers to the point with the largest  $y$  value possible on a function. A global (absolute) minimum refers to the point with the smallest  $y$  value possible. Together these two values are referred to as global extrema. There can only be one global maximum and only one global minimum. Global refers to entire space where the function is defined. Global extrema are also called absolute extrema.

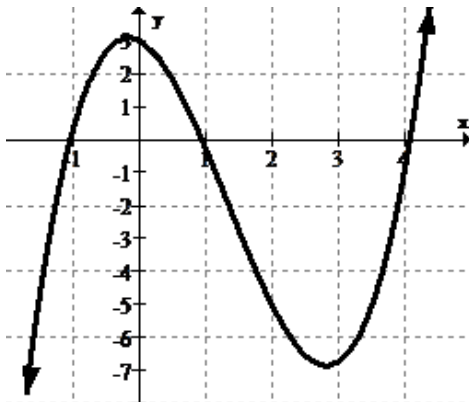
In addition to global maximums and global minimums, there are also local extrema or relative maximums and relative minimums. The word relative is used because in relation to some neighborhood, these values stand out as being the highest or the lowest.



Calculus uses advanced analytic tools to compute extreme values, but for the purposes of PreCalculus it is sufficient to be able to identify and categorize extreme values graphically or through the use of technology. For example, the TI-84 has a maximum finder when you select  $\langle 2^{nd} \rangle$  then  $\langle \text{trace} \rangle$ .

### Example A

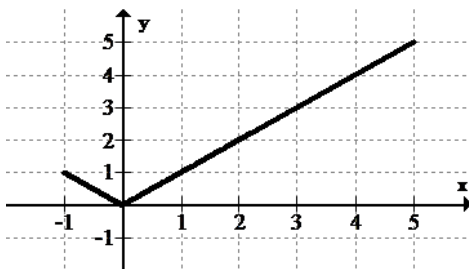
Identify and categorize all extrema:



**Solution:** Since the function appears from the arrows to increase and decrease beyond the display, there are no global extrema. There is a local maximum at approximately (0, 3) and a local minimum at approximately (2.8, -7).

### Example B

Identify and categorize all extrema:



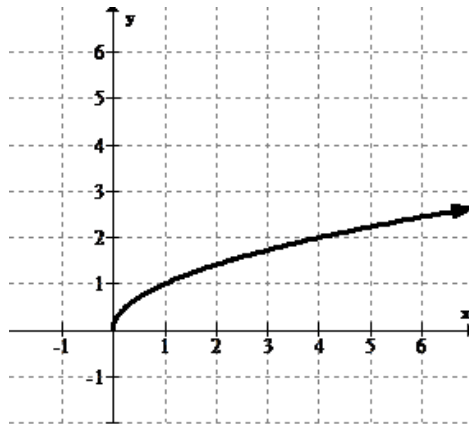


**Solution:** Since the function seems to abruptly end at the end points and does not go beyond the display, the endpoints are important.

There is a global minimum at  $(0, 0)$ . There is a local maximum at  $(-1, 1)$  and a global maximum at  $(5, 5)$ .

### Example C

Identify and categorize all extrema.



**Solution:** Since this function appears to increase to the right as indicated by the arrow there is no global maximum. There are not any other high points either, so there are no local maximums. There is only the end point at  $(0, 0)$  which is a global minimum.

### Concept Problem Revisited

Maximums and minimums should be intuitive because they simply identify the highest points and the lowest points, or the peaks and the valleys, in a graph. There is a formal distinction about whether a maximum is the highest on some local open interval (does not matter how small), or whether it is simply the highest overall.

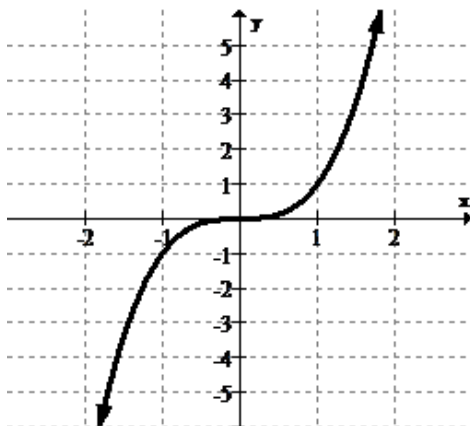
### Vocabulary

**Global extrema** and **absolute extrema** are synonyms that refer to the points with the  $y$  values that are either the highest or the lowest of the entire function.

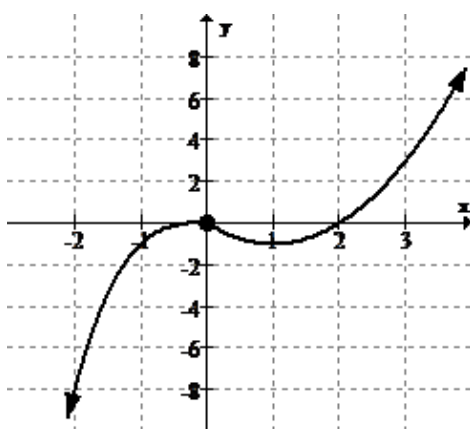
**Local extrema** and **relative extrema** are synonyms that refer to the points with the  $y$  values that are the highest or lowest of a local neighborhood of the function.

### Guided Practice

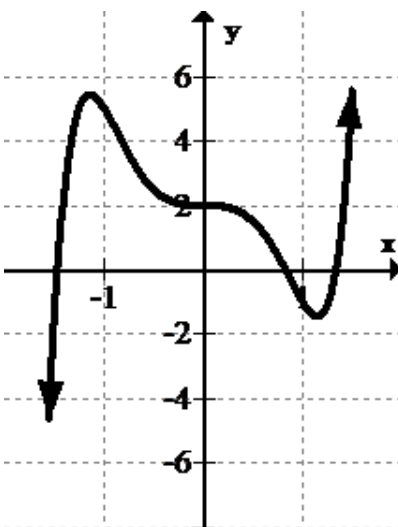
1. Identify and categorize all extrema.



2. Identify and categorize the extrema.



3. Identify and categorize the extrema.



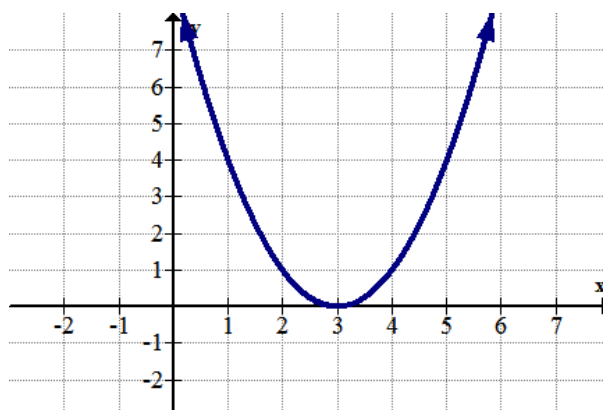
**Answers:**

1. There are no global or local maximums or minimums. The function flattens, but does not actually reach a peak or a valley.
2. There are no global extrema. There appears to be a local maximum at  $(0, 0)$  and a local minimum at  $(1, -1)$ .

3. There are no global extrema. There appears to be a local maximum at  $(-1.2, 5.3)$  and a local minimum at  $(1.2, -1.8)$ . These values are approximated. If a function was given, you would need to graph the function on a calculator and use the maximum and minimum features to identify more exact points.

### Practice

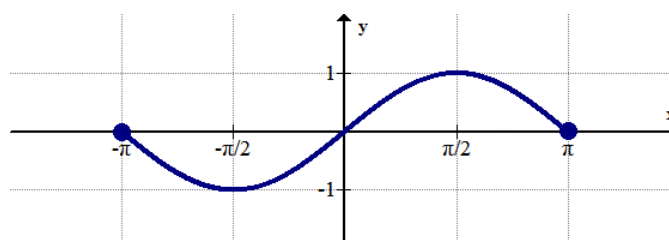
Use the graph below for 1-2.



1. Identify any global extrema.

2. Identify any local extrema.

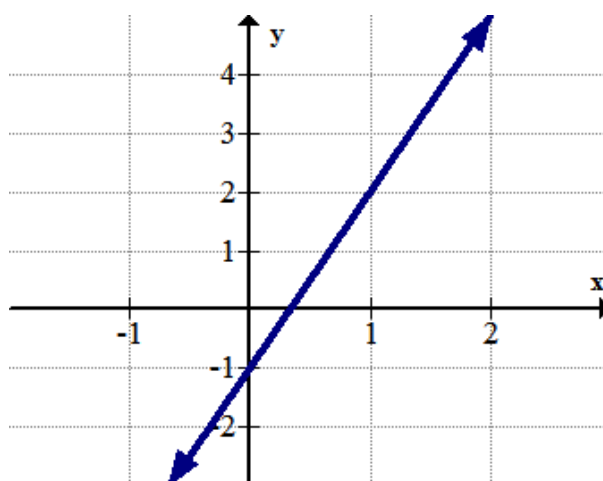
Use the graph below for 3-4.



3. Identify any global extrema.

4. Identify any local extrema.

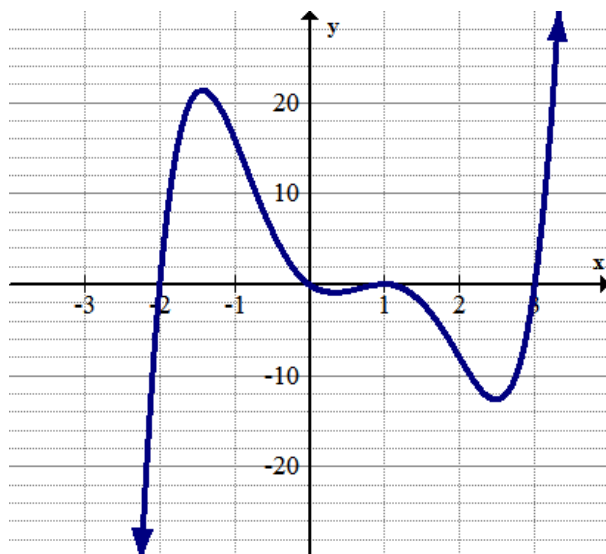
Use the graph below for 5-6.



5. Identify any global extrema.

6. Identify any local extrema.

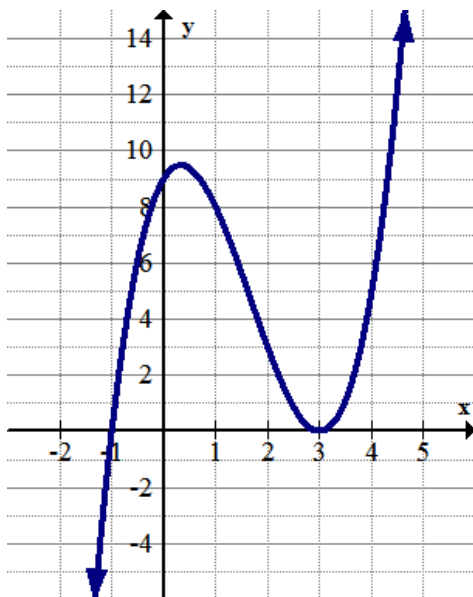
Use the graph below for 7-8.



7. Identify any global extrema.

8. Identify any local extrema.

Use the graph below for 9-10.



9. Identify any global extrema.

10. Identify any local extrema.

11. Explain the difference between a global maximum and a local maximum.

12. Draw an example of a graph with a global minimum and a local maximum, but no global maximum.

13. Draw an example of a graph with local maximums and minimums, but no global extrema.

14. Use your graphing calculator to identify and categorize the extrema of:

$$f(x) = \frac{1}{2}x^4 + 2x^3 - 6.5x^2 - 20x + 24.$$

15. Use your graphing calculator to identify and categorize the extrema of:

$$g(x) = -x^4 + 2x^3 + 4x^2 - 2x - 3.$$

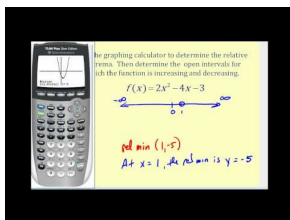
## 1.2 Increasing and Decreasing

Here you will apply interval notation to identify when functions are increasing and decreasing.

It is important to be able to distinguish between when functions are increasing and when they are decreasing. In business this could mean the difference between making money and losing money. In physics it could mean the difference between speeding up and slowing down.

How do you decide when a function is increasing or decreasing?

### Watch This



### MEDIA

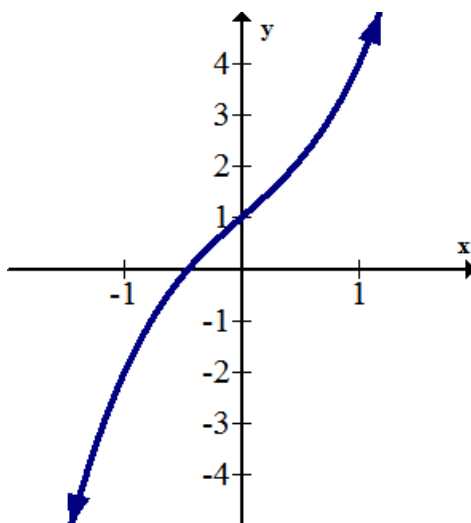
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URL: <http://www.ck12.org/flx/render/embeddedobject/57943>

<http://www.youtube.com/watch?v=78b4HOMVcKM> James Sousa: Determine Where a Function is Increasing or Decreasing

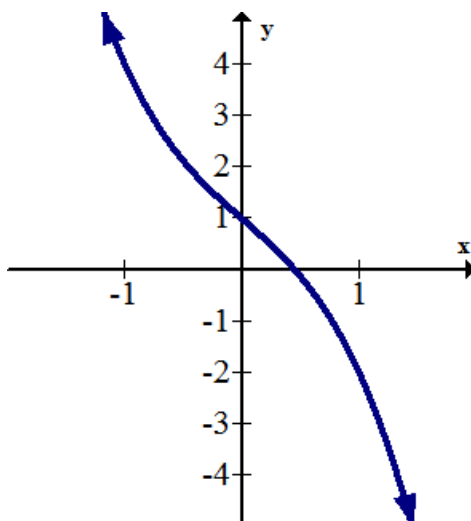
### Guidance

**Increasing** means places on the graph where the slope is positive.



**Decreasing** means places on the graph where the slope is negative. The formal definition of decreasing and strictly decreasing are identical to the definition of increasing with the inequality sign reversed.



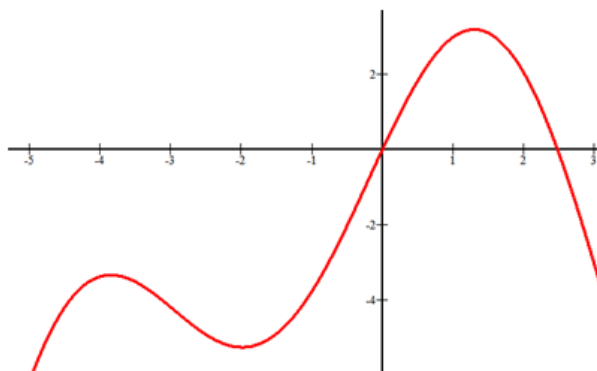


A function is called **monotonic** if the function only goes in one direction and never switches between increasing and decreasing.

Identifying analytically where functions are increasing and decreasing often requires Calculus. For PreCalculus, it will be sufficient to be able to identify intervals graphically and through your knowledge of what the parent functions look like.

### Example

Estimate the intervals where the function is increasing and decreasing.



### Solution:

Increasing:  $x \in (-\infty, -4) \cup (-2, 1.5)$

Decreasing:  $x \in (-4, -2) \cup (1.5, \infty)$

Note that open intervals are used because at  $x = -4, -2, 1.5$  the slope of the function is zero. This is where the slope transitions from being positive to negative. The reason why open parentheses are used is because the function is not actually increasing or decreasing at those specific points.

### Concept Problem Revisited

Increasing is where the function has a positive slope and decreasing is where the function has a negative slope. A common misconception is to look at the squaring function and see two curves that symmetrically increase away from zero. Instead, you should always read functions from left to right and draw slope lines and decide if they are positive or negative.

## Vocabulary

**Increasing** over an interval means to have a positive slope over that interval.

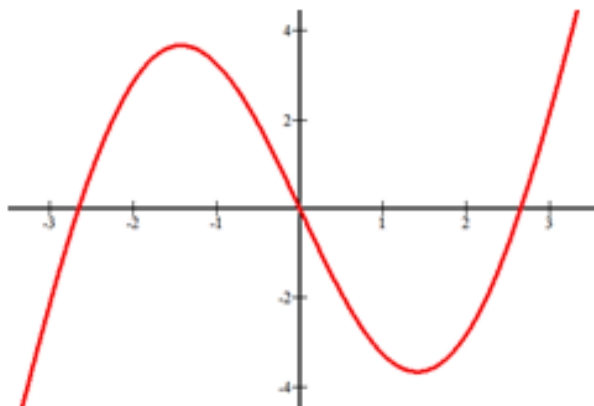
**Decreasing** over an interval means to have a negative slope over that interval.

**Monotonic** means that the function doesn't switch between increasing and decreasing at any point.

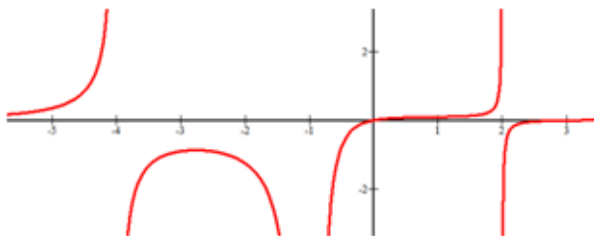
**Strictly** is an adjective that alters increasing and decreasing to exclude any flatness.

## Guided Practice

1. Estimate where the following function is increasing and decreasing.



2. Estimate where the following function is increasing and decreasing.



3. A continuous function has a global maximum at the point (3, 2), a global minimum at (5, -12) and has no relative extrema or other places with a slope of zero. What are the increasing and decreasing intervals for this function?

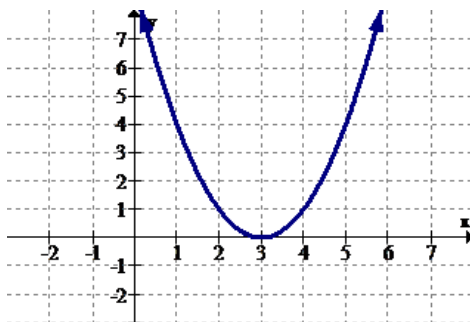
**Answers:**

1. Increasing:  $x \in (-\infty, -1.5) \cup (1.5, \infty)$ . Decreasing:  $x \in (-1.5, 1.5)$
2. Increasing  $x \in (-\infty, -4) \cup (-4, -2.7) \cup (-1, 2) \cup (2, \infty)$ . Decreasing  $x \in (-2.7, -1)$
3. Increasing  $x \in (-\infty, 3) \cup (5, \infty)$ . Decreasing  $x \in (3, 5)$

*Notice that the y coordinates are not used in the intervals. A common mistake is to want to use the y coordinates.*

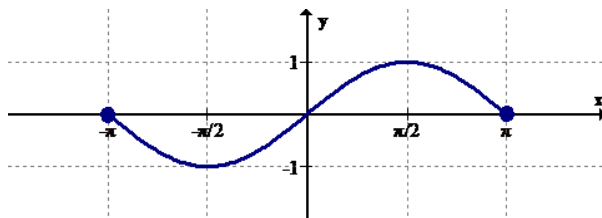
## Practice

Use the graph below for 1-2.



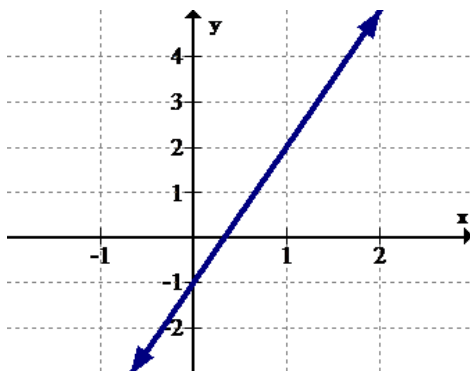
1. Identify the intervals (if any) where the function is increasing.
2. Identify the intervals (if any) where the function is decreasing.

Use the graph below for 3-4.



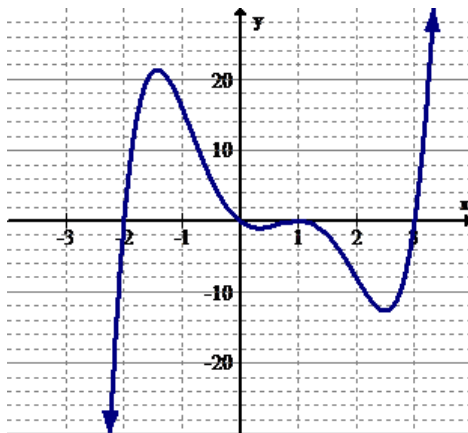
3. Identify the intervals (if any) where the function is increasing.
4. Identify the intervals (if any) where the function is decreasing.

Use the graph below for 5-6.



5. Identify the intervals (if any) where the function is increasing.
6. Identify the intervals (if any) where the function is decreasing.

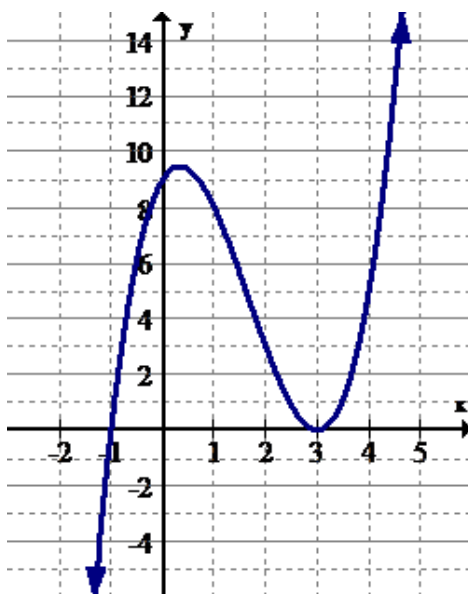
Use the graph below for 7-8.



7. Identify the intervals (if any) where the function is increasing.

8. Identify the intervals (if any) where the function is decreasing.

Use the graph below for 9-10.



9. Identify the intervals (if any) where the function is increasing.

10. Identify the intervals (if any) where the function is decreasing.

11. Give an example of a monotonically increasing function.

12. Give an example of a monotonically decreasing function.

13. A continuous function has a global maximum at the point  $(1, 4)$ , a global minimum at  $(3, -6)$  and has no relative extrema or other places with a slope of zero. What are the increasing and decreasing intervals for this function?

14. A continuous function has a global maximum at the point  $(1, 1)$  and has no other extrema or places with a slope of zero. What are the increasing and decreasing intervals for this function?

15. A continuous function has a global minimum at the point  $(5, -15)$  and has no other extrema or places with a slope of zero. What are the increasing and decreasing intervals for this function?

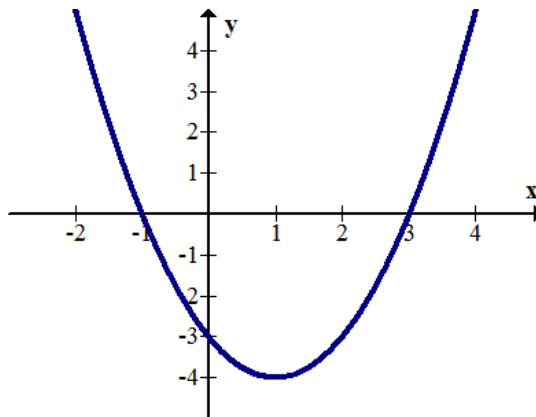


$x$ -intercepts are where functions cross the  $x$  axis and where the height of the function is zero. They are also called roots, solutions and zeroes of a function. They are found algebraically by setting  $y = 0$  and solving for  $x$ .

### Example A

What are the zeroes and  $y$ -intercepts of the parabola  $y = x^2 - 2x - 3$ ?

**Solution using Graph:**



The zeroes are at  $(-1, 0)$  and  $(3, 0)$ . The  $y$ -intercept is at  $(0, -3)$ .

**Solution using Algebra:**

Substitute 0 for  $y$  to find zeroes.

$$0 = x^2 - 2x - 3 = (x - 3)(x + 1)$$

$$y = 0, x = 3, -1$$

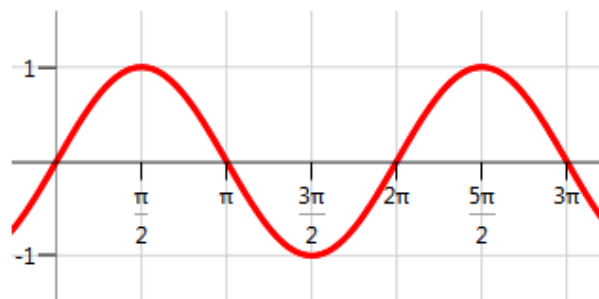
Substitute 0 for  $x$  to find the  $y$ -intercept.

$$y = (0)^2 - 2(0) - 3 = -3$$

$$x = 0, y = -3$$

### Example B

Identify the zeroes and  $y$ -intercepts for the sine function.



**Solution:** The  $y$ -intercept is  $(0, 0)$ . There are four zeroes visible on this portion of the graph. One thing you know about the sine graph is that it is periodic and repeats forever in both directions. In order to capture every  $x$ -intercept, you must identify a pattern instead of trying to write out every single one.

The visible  $x$ -intercepts are  $0, \pi, 2\pi, 3\pi$ . The pattern is that there is an  $x$ -intercept every multiple of  $\pi$  including negative multiples. In order to describe all of these values you should write:

The  $x$ -intercepts are  $\pm n\pi$  where  $n$  is an integer  $\{0, \pm 1, \pm 2, \dots\}$ .



**Example C**

Identify the intercepts and zeroes of the function:  $f(x) = \frac{1}{100}(x-3)^3(x+2)^2$ .

**Solution:**

To find the y-intercept, substitute 0 for x:

$$y = \frac{1}{100}(0-3)^3(0+2)^2 = \frac{1}{100}(-27)(4) = -\frac{108}{100} = -1.08$$

To find the x-intercepts, substitute 0 for y:

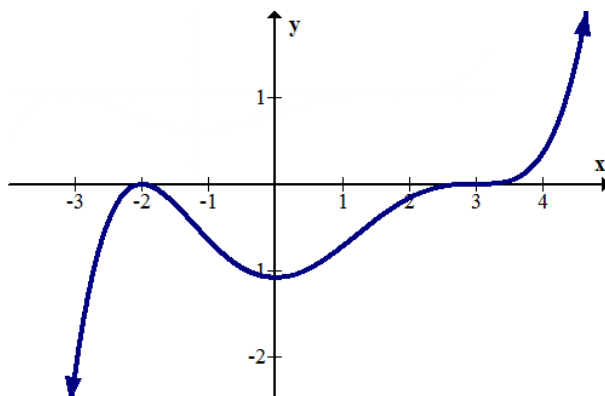
$$0 = \frac{1}{100}(x-3)^3(x+2)^2$$

$$x = 3, -2$$

Thus the y-intercept is (0, -1.08) and the x-intercepts are (3, 0) and (-2, 0).

**Concept Problem Revisited**

Graphically the function has zeroes at -2 and 3 with a y intercept at about -1.1. The algebraic solution is demonstrated in Example C.

**Vocabulary**

**Zeroes, roots, solutions** and **x-intercepts** are synonyms for the points where a function crosses the x axis.

A **y-intercept** is the point where a function crosses the y axis.

*Note that in order for a function to pass the vertical line test, it must only have one y-intercept, but it may have multiple x-intercepts.*

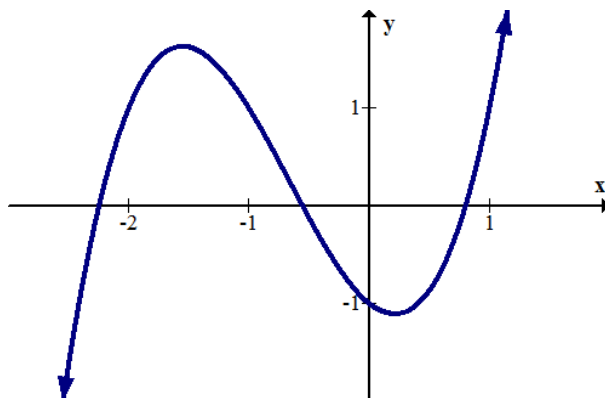
**Guided Practice**

1. Determine the zeroes and y-intercept of the following function using algebra:  $f(x) = (x+3)^2(x-2)$

2. Determine the roots and y-intercept of the following function using algebra or a graph:

$$f(x) = x^4 + 3x^3 - 7x^2 - 15x + 18$$

3. Determine the intercepts of the following function graphically.

**Answers:**

1. The y-intercept is (0, -18). The zeroes (x-intercepts) are (-3, 0) and (2, 0).
2. The y-intercept is (0, 18). The roots (x-intercepts) are (2, 0), (1, 0) and (-3, 0).
3. The y-intercept is approximately (0, -1). The x-intercepts are approximately (-2.3, 0), (-0.4, 0) and (0.7, 0). When finding values graphically, answers are always approximate. Exact answers need to be found analytically.

**Practice**

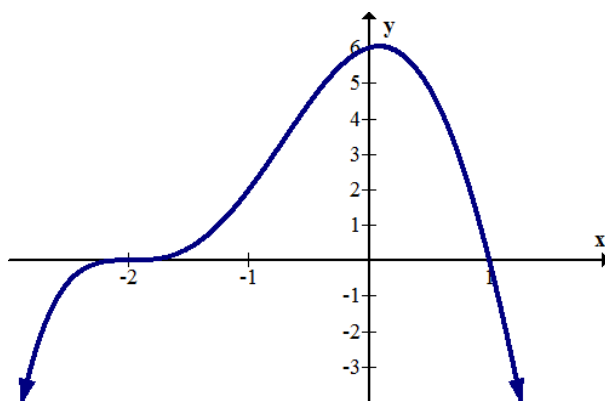
1. Determine the zeroes and y-intercept of the following function using algebra:

$$f(x) = (x + 1)^3(x - 4)$$

2. Determine the roots and y-intercept of the following function using algebra or a graph:

$$g(x) = x^4 - 2x^3 - 7x^2 + 20x - 12$$

3. Determine the intercepts of the following function graphically:



Find the intercepts for each of the following functions.

4.  $y = x^2$
5.  $y = x^3$
6.  $y = \ln(x)$
7.  $y = \frac{1}{x}$
8.  $y = e^x$
9.  $y = \sqrt{x}$

10. Are there any functions without a  $y$ -intercept? Explain.
11. Are there any functions without an  $x$ -intercept? Explain.
12. Explain why it makes sense that an  $x$ -intercept of a function is also called a “zero” of the function.

Determine the intercepts of the following functions using algebra or a graph.

13.  $h(x) = x^3 - 6x^2 + 3x + 10$

14.  $j(x) = x^2 - 6x - 7$

15.  $k(x) = 4x^4 - 20x^3 - 3x^2 + 14x + 5$

Key features of functions were explored through the use of ten basic function families. Transforming functions and finding inverses of functions were also considered.

---

## 1.4 References

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## CONCEPT

## 2

# Graphing Equations and Word Problems

Solving Equations Graphically:

**TABLE 2.1:**

	When you kick a soccer ball (or shoot an arrow, fire a missile or throw a stone) it arcs up into the air and comes down again ... ... following the path of a parabola!
--	--

To solve an equation means to find all the values that make the statement true.

To solve an equation graphically, draw the graph for each side, member, of the equation and see where the curves cross, are equal. The x values of these points, are the solutions to the equation.

There are many possible outcomes when one solves an equation.

<http://www.youtube.com/watch?v=h6MS4rFOX-s>

An example of a **Quadratic Equation**:

[Quadratic equations](#) pop up in many real world situations!

If you throw a ball (or shoot an arrow, fire a missile or throw a stone) it will go up into the air, slowing down as it goes, then come down again ...

... and a [Quadratic Equation](#) tells you where it will be!

## Quadratic Word Problems

View this video to see an example of a quadratic word problem:

<http://www.educrations.com/lesson/view/quadratic-word-problems/10791860/?ref=appemail>

## CHAPTER

## 3

# Polynomials and Rational Functions

## Chapter Outline

---

- 3.1**    **POLYNOMIAL FUNCTIONS**
  - 3.2**    **RATIONAL EXPRESSIONS**
  - 3.3**    **HOLES IN RATIONAL FUNCTIONS**
  - 3.4**    **ZEROES OF RATIONAL FUNCTIONS**
  - 3.5**    **VERTICAL ASYMPTOTES**
  - 3.6**    **HORIZONTAL ASYMPTOTES**
  - 3.7**    **REFERENCES**
- 

Here you will deepen your knowledge about polynomials from your work in algebra. You will review factoring and learn more advanced factoring techniques. You will learn how to divide polynomials and how polynomials and rational expressions are related. You will then focus on rational functions and learn about their unique features including three different types of asymptotes. Finally, you will put it all together and learn how to graph rational functions by hand.

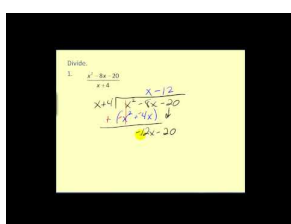


## 3.1 Polynomial Functions

Here you will learn how to perform long division with polynomials. You will see how synthetic division abbreviates this process. In addition to mastering this procedure, you will see how the remainder root theorem and the rational root theorem operate.

While you may be experienced in factoring, there will always be polynomials that do not readily factor using basic or advanced techniques. How can you identify the roots of these polynomials?

### Watch This

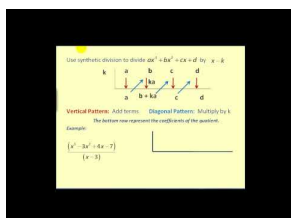


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URL: <http://www.ck12.org/flx/render/embeddedobject/60732>

<http://www.youtube.com/watch?v=brpNxPAkv1c> James Sousa: Dividing Polynomials-Long Division



#### MEDIA

Click image to the left or use the URL below.

URL: <http://www.ck12.org/flx/render/embeddedobject/60734>

<http://www.youtube.com/watch?v=5dBAAdzl2Mns> James Sousa: Dividing Polynomials-Synthetic Division

**Definition:** The degree of a polynomial is the highest exponent of the polynomial.

**Definition:** The constant of a polynomial is the term of the polynomial without an x.

Ex) The degree and the constant of the polynomial function

$$f(x) = 4x^3 - x^2 + 10x - 11$$

$$\text{Degree} = 3 \quad \text{Constant} = -11$$

### YOUR TURN

Find the degree and the constant of the following:

$$f(x) = x^2 + 3x + 2$$

$$g(x) = 3x^4 - x^7 + 5 + 4x^5$$

$$h(x) = x - 1$$

$$f(x) = x^2 + 3x + 2$$

Degree = 2      Constant = 2

$$g(x) = 3x^4 - x^7 + 5 + 4x^5$$

Degree = 7      Constant = 5

$$3) h(x) = x - 1$$

Degree = 1      Constant = - 1

Definition: The **lead coefficient** of a polynomial function is the coefficient of the variable with the highest exponent.

$$\text{Ex) } f(x) = 3x^2 - 5x^3 + 4$$

The lead coefficient is  $- 5$ .

### YOUR TURN

Find the lead coefficient of:

$$\text{a) } f(x) = 3x^2 + 3x + 2$$

$$\text{b) } g(x) = 3x^4 - x^7 + 5 + 4x^5$$

$$\text{c) } h(x) = x - 1$$

- a) 3
- b) -1
- c) 1

Find the lead coefficients of:

$$f(x) = x^2 + 3x + 2$$

$$g(x) = 3x^4 - x^7 + 5 + 4x^5$$

$$h(x) = x - 1$$

$$f(x) \rightarrow 1$$

$$g(x) \rightarrow -1$$

$$h(x) \rightarrow 1$$

## SYNTHETIC DIVISION

$$5x^2 - 17x - 12 \text{ divided by } x - 4$$

To use synthetic division, follow the following steps:

- 1) Put the numerator in descending order if not already.

$$5x^2 - 17x - 12$$

- 2) Write the coefficients of the numerator without the x's.

$$5x^2 - 17x - 12$$

$$5 \quad -17 \quad -12$$

- 3) Put the denominator in the form

$$x - (\text{a number})$$

$$x - (4)$$

4) Take the number write it before the coefficients of the numerator on the outside of an upside down division symbol.

$$\begin{array}{r|rrrr} 4 & 5 & -17 & -12 & \\ & & & & \end{array}$$

-

5) Rule: Add down,  
multiply across.

$$\begin{array}{r|rrrr} 4 & 5 & -17 & -12 & \\ & & & & \\ \hline & 5 & & & \end{array}$$

$$\begin{array}{r|rrrr} 4 & 5 & -17 & -12 & \\ & & & & \\ \hline & & 20 & & \\ & 5 & & & \end{array}$$

$$\begin{array}{r|rrrr} 4 & 5 & -17 & -12 & \\ & & & & \\ \hline & & 20 & & \\ & 5 & & 3 & \end{array}$$

$$\begin{array}{r|rrrr} 4 & 5 & -17 & -12 & \\ & & & & \\ \hline & & 20 & 12 & \\ & 5 & & 3 & \end{array}$$

$$\begin{array}{r|rrrr} 4 & 5 & -17 & -12 & \\ & & & & \\ \hline & & 20 & 12 & \\ & 5 & 3 & 0 & \end{array}$$

5      3      0    is the answer.

**Step 6) The answer is always one less degree than the original numerator .**

Definition: The degree of a polynomial is the highest exponent of the polynomial.

$$\frac{5x^2 - 17x - 12}{x - 4} \leftarrow \text{degree 2}$$

5      3      0       $\rightarrow$  degree 1

So 5 stands for  $x^1$ .

5      3      0  
 $x^1$      $x^0$     remainder

$$5x^1 + 3x^0$$

$$x^0 = 1$$

**$5x + 3$**  is the answer.

For more help with synthetic division, WATCH THIS:

Synthetic Division

<http://www.educreations.com/lesson/view/synthetic-division/11572367/?ref=appemail>

**SYNTHETIC DIVISION WHEN THE NUMERATOR IS MISSING TERMS**

$$\frac{x^4 - 10x^2 - 2x + 4}{x + 3}$$

$$x^4 - 10x^2 - 2x + 4$$

Notice there is no  $x^3$  term in the numerator

So when you set up the numerator's coefficients, you must include a zero for the  $x^3$  term.

**RULE:** You must always fill in a zero for every missing term in the numerator.

$x^4 - 10x^2 - 2x + 4$  becomes

$$x^4 + 0x^3 - 10x^2 - 2x + 4$$

$$1 \quad 0 \quad -10 \quad -2 \quad 4$$

$$\begin{array}{r} x^4 - 10x^2 - 2x + 4 \\ x + 3 \end{array} \rightarrow x - (-3)$$

$$\begin{array}{r} -3 \mid 1 \quad 0 \quad -10 \quad -2 \quad 4 \\ \hline \end{array}$$

Remember: Add down, multiply across.

$$\begin{array}{r} -3 \mid 1 \quad 0 \quad -10 \quad -2 \quad 4 \\ \hline \quad -3 \quad 9 \quad 3 \quad -3 \\ \hline 1 \quad -3 \quad -1 \quad 1 \quad 1 \end{array}$$

$$\begin{array}{r} x^4 - 10x^2 - 2x + 4 \\ x + 3 \end{array} \text{ is degree 4}$$

$$1 \quad -3 \quad -1 \quad 1 \quad 1 \leftarrow \text{degree 3}$$

$$\begin{array}{r} 1 \quad -3 \quad -1 \quad 1 \quad 1 \\ x^3 \quad x^2 \quad x^1 \quad x^0 \quad \text{remainder} \end{array}$$

$$x^3 - 3x^2 - x + 1 + \frac{1}{x+3}$$

Watch this video to learn how to analyze polynomial function graphs:

Analyzing Graphs

<http://www.educreations.com/lesson/view/analyzing-graphs-lead-coefficient-test/11092707/?ref=appemail>



To compile a list of possible zeros (x-intercepts), use the rational zero test.

### RATIONAL ZERO TEST

Every rational zero (non imaginary or non-irrational) of a polynomial function is in the form

$$P = \pm \text{Factors of the constant}$$

$$Q = \pm \text{Factors of the lead coef.}$$

So to make a list of possible rational zeros of

$$f(x) = 2x^4 + 5x^3 - 15x^2 - 45x - 27$$

1) Set up the fraction

$$\text{Constant} \rightarrow 27$$

$$\text{Lead Coefficient} \rightarrow 2$$

Note: Ignore the signs of the constant and lead coefficient.

2) Write out the factors of the constant and the lead coefficient.

$$\frac{27}{2} =$$

$$\frac{27}{2} = \frac{1, 3, 9, 27}{1, 2}$$

3) Write out all of the combinations possible with the numerator and the denominator.

$$\frac{1, 3, 9, 27}{1, 2}$$

$$\frac{1}{1}, \frac{3}{1}, \frac{9}{1}, \frac{27}{1}, \frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \frac{27}{2}$$

4) Simplify and eliminate any equivalent answers

(Here none of the fractions are equivalent)

1, 3, 9, 27,  $\frac{1}{2}$ ,  $\frac{3}{2}$ ,  $\frac{9}{2}$ ,  $\frac{27}{2}$

5) Write a  $\pm$  in front of each possible zero.

$\pm 1, \pm 3, \pm 9, \pm 27, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{9}{2}, \pm \frac{27}{2}$

So the possible rational zeros of

**$f(x) = 2x^4 + 5x^3 - 15x^2 - 45x - 27$  are:**

$\pm 1, \pm 3, \pm 9, \pm 27, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{9}{2}, \pm \frac{27}{2}$

Note: This is a list of possible rational zeros. There can be irrational zeros or imaginary zeros. This list does not tell you the irrational or imaginary zeros. In order to see which from the list are ACTUAL x-intercepts is to graph the function and see where the graph crosses the x-axis.

**WATCH THIS VIDEO FOR MORE ON THE RATIONAL ZERO TEST:**

**Rational Zero Test**

**<http://www.educreations.com>**

## DESCARTES RULE OF SIGNS

-

### PART 1

The number of positive zeros of a function is equal to the number of sign changes of the function or is less than that by an even number.

$$f(x) = x^3 - 4x^2 - 4x + 16$$

**How many positive zeros can  $f(x)$  have ?**

Look at the signs of the terms.

$$x^3 \quad - 4x^2 \quad - 4x \quad + 16$$

+       -       -       +

**How many sign changes ?**

+       -       -       +

/       /       /

yes    no    yes

**There are 2 sign changes.**

“or is less than that by an even number.”

Take the number of sign changes and keep subtracting 2 from it until you get to 0 .

$$2 - 2 = 0$$

So there could be either 2 or 0 positive zeros of

$$x^3 - 4x^2 - 4x + 16.$$

“The number of positive zeros of a function is equal to the number of sign changes of the function”

### PART 2 OF DESCARTES RULE OF SIGNS

The number of negative zeros of a function is equal to the number of sign changes of  $f(-1x)$  or is less than that by a even number.

Ex) Find the number of negative zeros of

$$f(x) = x^3 - 4x^2 - 4x + 16$$

First you have to find  $f(-1x)$ .

So in  $f(x)$  everywhere there is an  $x$ , substitute in  $(-1x)$ .

$$f(x) = x^3 - 4x^2 - 4x + 16$$

$$f(-1x) = (-1x)^3 - 4(-1x)^2 - 4(-1x) + 16$$

now simplify:

$$f(-1x) = (-1x)^3 - 4(-1x)^2 - 4(-1x) + 16$$

$$= (-1)^3 x^3 - 4(-1)^2 x^2 + 4x + 16$$

$$= -1x^3 - 4(+1) x^2 + 4x + 16$$

$$= -1x^3 - 4x^2 + 4x + 16$$

$$\text{So } f(-1x) = -1x^3 - 4x^2 + 4x + 16$$

How many sign changes ?

$$-1x^3 - 4x^2 + 4x + 16$$

$$- \quad - \quad + \quad +$$

$$/ \quad / \quad /$$

no    yes    no

There 1 sign change so there is definitely 1 negative zero.

**Note:** Descartes Rule of Signs gives you a definite answer when there is 1 sign change or no sign changes.

Find the number of positive and negative zeros of:

$$f(x) = 2x^4 + 5x^3 - 15x^2 - 45x - 27$$

Now try one: Find the number of possible POSITIVE and the number of possible NEGATIVE zeros of:

$$f(x) = 2x^4 + 5x^3 - 15x^2 - 45x - 27$$

### Number of positive zeros

Look at the Sign changes of the original equation:

$$f(x) = 2x^4 + 5x^3 - 15x^2 - 45x - 27$$

+	+	-	-	-
/	/	/	/	
no	yes	no	no	

1 sign change means definitely

1 positive zero.

**Number of negative zeros**

**Find the number of sign changes of  $f(-1x)$ .**

$$f(x) = 2x^4 + 5x^3 - 15x^2 - 45x - 27$$

$$f(-1x) = 2(-1x)^4 + 5(-1x)^3 - 15(-1x)^2 - 45(-1x) - 27$$

$$= 2(1x^4) + 5(-1x^3) - 15(1x^2) + 45x - 27$$

$$= 2x^4 - 5x^3 - 15x^2 + 45x - 27$$

$$f(-1x) = 2x^4 - 5x^3 - 15x^2 + 45x - 27$$

Number of sign changes

+	-	-	+	-
/	/	/	/	
yes	no	yes	yes	

3 sign changes means either

3 or 1 sign negative zeros

$$f(x) = 2x^4 + 5x^3 - 15x^2 - 45x - 27$$

So there is definitely 1 positive zero and either 3 or 1 negative zeros.

To check, graph the function and see where the graph crosses the x-axis.

FOR MORE ON DESCARTES RULE OF SIGNS, watch this: Descartes Rule of Signs <http://www.educreations.com>.

[com/lesson/view/descartes-rule-of-signs/12494070/?ref=appemail](https://www.ck12.org/lesson/view/descartes-rule-of-signs/12494070/?ref=appemail)

Here is a video on how we are going to graph polynomials: Graphing Polynomials <https://www.educreations.com/lesson/view/graphing-polynomials/24589433/?ref=app>

## 3.2 Rational Expressions

Here you will add, subtract, multiply and divide rational expressions in order to help you solve and graph rational expressions in the future.

A rational expression is a ratio just like a fraction. Instead of a ratio between numbers, a rational expression is a ratio between two expressions.

### Graphing Rational Functions

-

Def: A RATIONAL FUNCTION is a function that is in the form of a fraction.

$$\text{Ex) } f(x) = \frac{x^2 - 3x - 4}{x^2 - 4}$$

We are going to learn how to graph these functions.

Are there any numbers that if we plugged into

$$f(x) = \frac{x^2 - 3x - 4}{x^2 - 4}$$

would cause a problem ?

$$x = \pm 2$$

If you plugged in either + 2 or - 2 into

$$f(x) = \frac{x^2 - 3x - 4}{x^2 - 4}$$

the denominator would be 0 and that's a problem.

Def: An asymptote is a dotted line which is not part of the function but gives the function structure.

The function will get closer and closer to the asymptotes, but will never cross the asymptote.

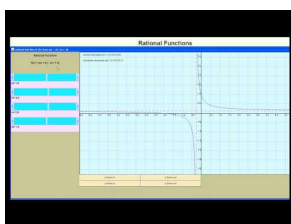
## 3.3 Holes in Rational Functions

Here you will start factoring rational expressions that have holes known as removable discontinuities.

In a function like  $f(x) = \frac{(3x+1)(x-1)}{(x-1)}$ , you should note that the factor  $(x-1)$  clearly cancels leaving only  $3x+1$ . This appears to be a regular line. What happens to this line at  $x=1$ ?

### Watch This

Watch the first part of this video. Focus on how to identify the holes.



### MEDIA

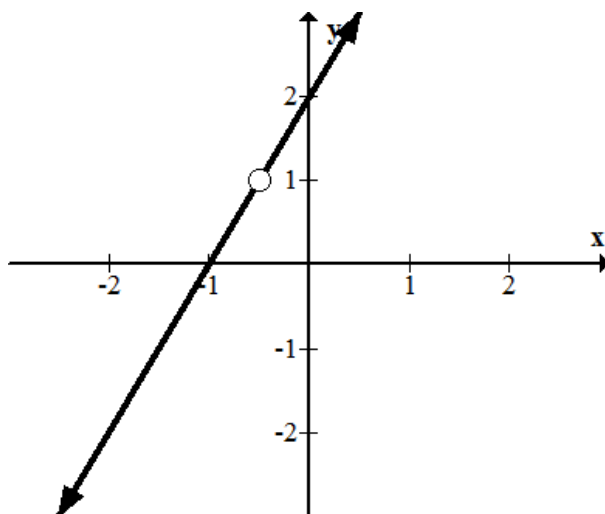
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URL: <http://www.ck12.org/flx/render/embeddedobject/60822>

<http://www.youtube.com/watch?v=OEQnQNvJtG0> James Sousa: Graphing Rational Functions

### Guidance

A hole on a graph looks like a hollow circle. It represents the fact that the function approaches the point, but is not actually defined on that precise  $x$  value.



The reason why this function is not defined at  $-\frac{1}{2}$  is because  $-\frac{1}{2}$  is not in the domain of the function.

$$f(x) = (2x+2) \cdot \frac{(x+\frac{1}{2})}{(x+\frac{1}{2})}$$

As you can see,  $f(-\frac{1}{2})$  is undefined because it makes the denominator of the rational part of the function zero which makes the whole function undefined. Also notice that once the factors are canceled/removed then you are left with a normal function which in this case is  $2x+2$ .



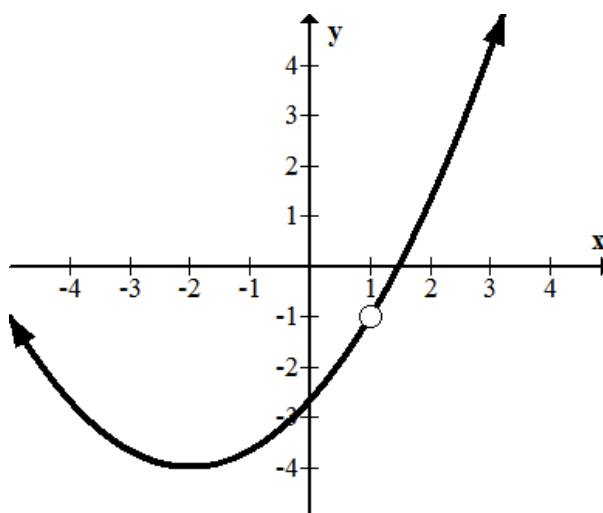
This is the essence of dealing with holes in rational functions. You should cancel what you can and graph the function like normal making sure to note what  $x$  values make the function undefined. Once the function is graphed without holes go back and insert the hollow circles indicating what  $x$  values are removed from the domain. This is why holes are called removable discontinuities.

### Example A

Graph the following rational function and identify any removable discontinuities.

$$f(x) = \left(\frac{1}{3}(x+2)^2 - 4\right) \cdot \frac{(x-1)}{(x-1)}$$

**Solution:** This function is already separated into two parts, the rational part and a parabola. To graph the function, simply graph the parabola and then insert a hollow circle at the appropriate height at  $x = 1$ .



The hole is at  $(1, -1)$  because after removing the factors that cancel,  $f(1) = -1$ .

Note that most problems will require significant algebraic steps to reach this point. This example emphasizes that the backbone of the function is essentially a parabola with only one difference.

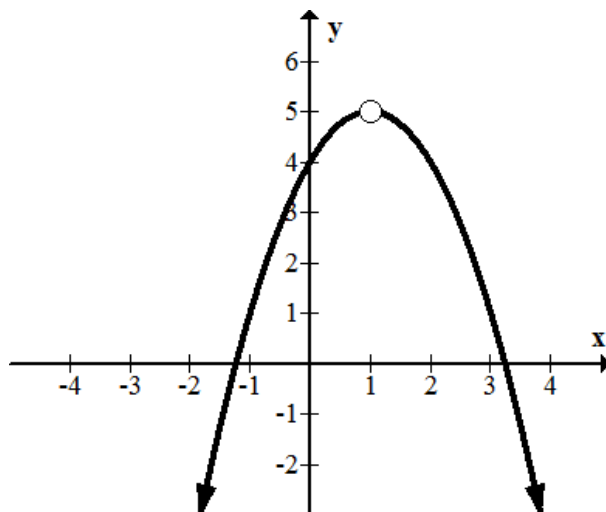
### Example B

Graph the following rational function and identify any removable discontinuities.

$$f(x) = \frac{-x^3 + 3x^2 + 2x - 4}{x-1}$$

**Solution:** This function requires some algebra to change it so that the removable factors become obvious. You should suspect that  $(x-1)$  is a factor of the numerator and try polynomial or synthetic division to factor. When you do, the function becomes:

$$f(x) = \frac{(-x^2 + 2x + 4)(x-1)}{(x-1)}$$



The removable discontinuity occurs at (1, 5).

### Example C

Graph the following rational function and identify any removable discontinuities.

$$f(x) = \frac{x^6 - 6x^5 + 5x^4 + 27x^3 - 48x^2 - 9x + 54}{x^3 - 7x - 6}$$

**Solution:** This is probably one of the most challenging rational expressions with only holes that people ever try to graph by hand. There are multiple ways to start, but a good habit to get into is to factor everything you possibly can initially. The denominator seems less complicated with possible factors  $(x \pm 1)$ ,  $(x \pm 2)$ ,  $(x \pm 3)$ ,  $(x \pm 6)$ . Using polynomial division, you will find the denominator becomes:

$$f(x) = \frac{x^6 - 6x^5 + 5x^4 + 27x^3 - 48x^2 - 9x + 54}{(x+1)(x+2)(x-3)}$$

The factors of the denominator are strong hints as to the factors of the numerator so use polynomial division and try each. When you fully factor the numerator you will have:

$$f(x) = \frac{(x^3 - 6x^2 + 12x - 9)(x+1)(x+2)(x-3)}{(x+1)(x+2)(x-3)}$$

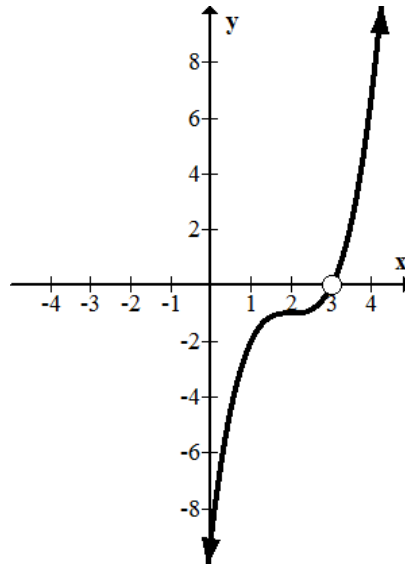
Note the factors that cancel  $(x = -1, -2, 3)$  and then work with the cubic function that remains.

$$f(x) = x^3 - 6x^2 + 12x - 9$$

At this point it is probably reasonable to make a table and plot points to get a sense of where this cubic function lives. You also could notice that the coefficients are almost of the pattern 1 3 3 1 which is the binomial expansion. By separating the -9 into -8 -1 you can factor the first four terms.

$$f(x) = x^3 - 6x^2 + 12x - 8 - 1 = (x - 2)^3 - 1$$

This is a cubic function that has been shifted right by two units and down one unit.



Note that there are two holes that do not fit in the graph window. When this happens you still need to note where they would appear given a properly sized window. To do this, substitute the invalid  $x$  values:  $x = -1, -2, 3$  into the factored cubic that remained after canceling.

$$f(x) = (x - 2)^3 - 1$$

Holes:  $(3, 0)$ ;  $(-1, -28)$ ;  $(-2, -65)$

### Concept Problem Revisited

$$f(x) = \frac{(3x+1)(x-1)}{(x-1)}$$

For this function that is not defined at  $x = 1$  there is a removable discontinuity that is represented as a hollow circle on the graph. Otherwise the function behaves precisely as  $3x + 1$ .

### Vocabulary

A **removable discontinuity**, also known as a hole, is a point on a function that occurs because a factor can be canceled from the numerator and the denominator of the rational function.

A **rational function** is a function with at least one rational expression.

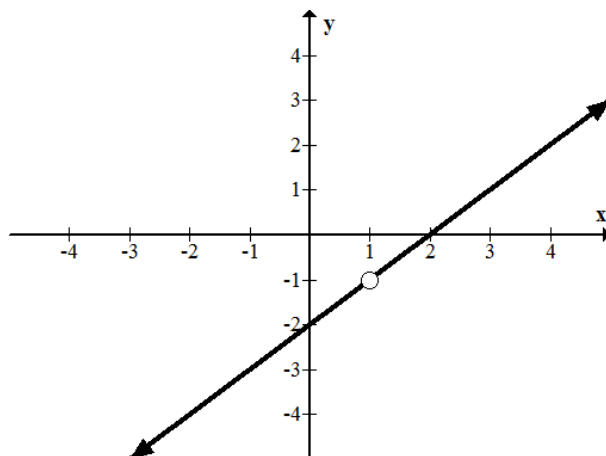
A **rational expression** is a ratio of two polynomial expressions.

### Guided Practice

1. Without graphing, identify the location of the holes of the following function.

$$f(x) = \frac{x^3 + 4x^2 + x - 6}{x^2 + 5x + 6}$$

2. What is a possible equation for the following rational function?



3. Identify the holes of the following function.

$$f(x) = x \cdot \frac{\sin x}{\sin x}$$

**Answers:**

1. First factor everything. Then, identify the  $x$  values that make the denominator zero and use those values to find the exact location of the holes.

$$f(x) = \frac{(x+2)(x+3)(x-1)}{(x+3)(x+2)}$$

Holes:  $(-3, -4)$ ;  $(-2, -3)$

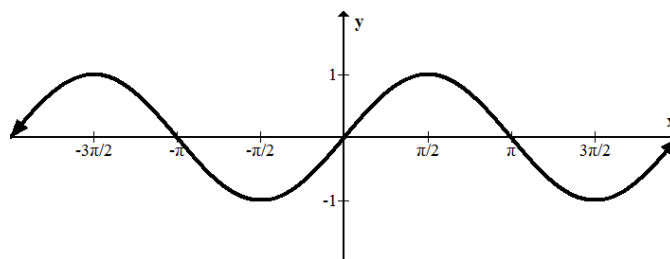
2. The function seems to be a line with a removable discontinuity at  $(1, -1)$ . The line has slope 1 and y-intercept of  $-2$  and so has the equation:

$$f(x) = x - 2$$

The removable discontinuity must not allow the  $x$  to be 1 which implies that it is of the form  $\frac{x-1}{x-1}$ . Therefore, the function is:

$$f(x) = \frac{(x-2)(x-1)}{x-1}$$

3. While the function is not a rational function because it includes a trigonometric expression, the exact same tools apply. You should ask yourself: when is the sine function equal to zero? Since the sine function is one of the basic functions you can sketch the function and note that it has a height of 0 at  $0, \pm\pi, \pm2\pi \dots$



Since the function is just the line  $f(x) = x$  with holes everywhere the sine function is zero, there are an infinite number of holes. The holes occur at  $(0, 0), (\pi, \pi), (-\pi, -\pi), (2\pi, 2\pi) \dots$

## Practice

1. How do you find the holes of a rational function?
2. What's the difference between a hole and a removable discontinuity?

3. If you see a hollow circle on a graph, what does that mean?

Without graphing, identify the location of the holes of the following functions.

4.  $f(x) = \frac{x^2+3x-4}{x-1}$

5.  $g(x) = \frac{x^2+8x+15}{x+3}$

6.  $h(x) = \frac{x^3+6x^2+2x-8}{x^2+x-2}$

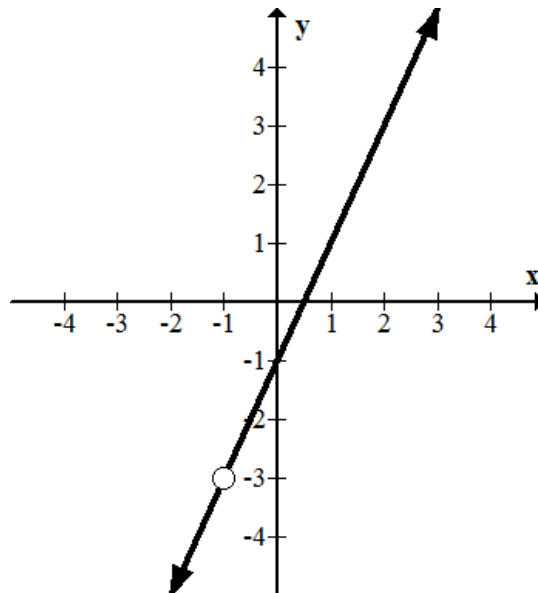
7.  $k(x) = \frac{x^3+6x^2+2x-8}{x^2-3x+2}$

8.  $j(x) = \frac{x^3+4x^2-17x-60}{x^2-9}$

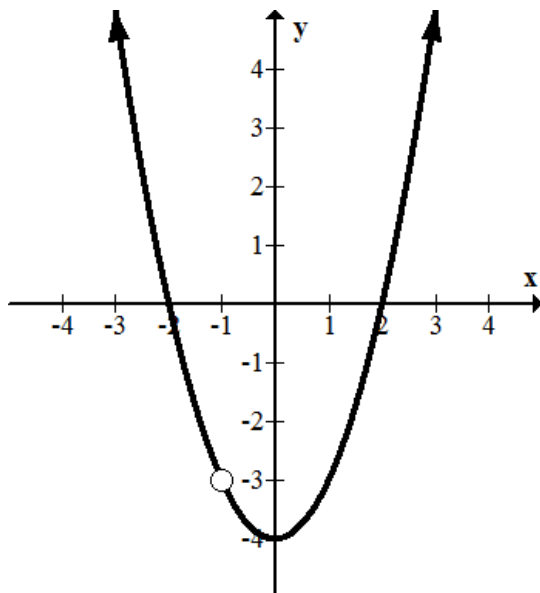
9.  $f(x) = \frac{x^3+4x^2-17x-60}{x^2-5x+4}$

10.  $g(x) = \frac{x^3-4x^2-19x-14}{x^2-8x+7}$

11. What is a possible equation for the following rational function?



12. What is a possible equation for the following rational function?



Sketch the following rational functions.

13.  $f(x) = \frac{x^3 + 4x^2 - 17x - 60}{x^2 - x - 12}$

14.  $g(x) = \frac{x^3 + 4x^2 - 17x - 60}{x^2 + 8x + 15}$

15.  $h(x) = \frac{x^3 - 4x^2 - 19x - 14}{x^2 - 6x - 7}$

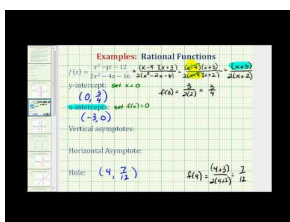
## 3.4 Zeroes of Rational Functions

Here you will revisit how to find zeroes of functions by focusing on rational expressions and what to do in special cases where the zeroes and holes seem to overlap.

The zeroes of a function are the collection of  $x$  values where the height of the function is zero. How do you find these values for a rational function and what happens if the zero turns out to be a hole?

### Watch This

Focus on the portion of this video discussing holes and  $x$ -intercepts.



### MEDIA

Click image to the left or use the URL below.

URL: <http://www.ck12.org/flx/render/embeddedobject/60830>

<http://www.youtube.com/watch?v=UnVZs2EaEjI> James Sousa: Find the Intercepts, Asymptotes, and Hole of a Rational Function

### Guidance

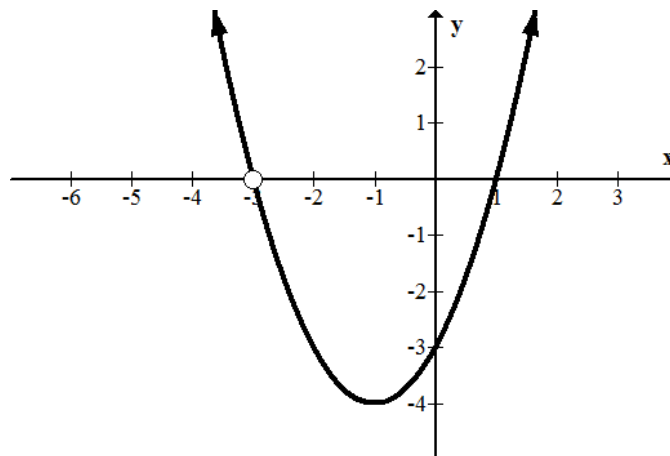
Zeroes are also known as  $x$ -intercepts, solutions or roots of functions. They are the  $x$  values where the height of the function is zero. For rational functions, you need to set the numerator of the function equal to zero and solve for the possible  $x$  values. If a hole occurs on the  $x$  value, then it is not considered a zero because the function is not truly defined at that point.

### Example A

Identify the zeroes and holes of the following rational function.

$$f(x) = \frac{(x-1)(x+3)(x+3)}{x+3}$$

**Solution:** Notice how one of the  $x+3$  factors seems to cancel and indicate a removable discontinuity. Even though there are two  $x+3$  factors, the only zero occurs at  $x = 1$  and the hole occurs at  $(-3, 0)$ .



### Example B

Identify the zeroes, holes and y intercepts of the following rational function without graphing.

$$f(x) = \frac{x(x-2)(x-1)(x+1)(x+2)}{(x-1)(x+1)}$$

**Solution:** The holes occur at  $x = -1, 1$ . To get the exact points, these values must be substituted into the function with the factors canceled.

$$\begin{aligned} f(x) &= x(x-2)(x+1)(x+2) \\ f(-1) &= 0, f(1) = -6 \end{aligned}$$

The holes are  $(-1, 0); (1, 6)$ . The zeroes occur at  $x = 0, 2, -2$ . The zero that is supposed to occur at  $x = -1$  has already been demonstrated to be a hole instead.

### Example C

Create a function with holes at  $x = 1, 2, 3$  and zeroes at  $x = 0, 4$ .

**Solution:** There are an infinite number of possible functions that fit this description because the function can be multiplied by any constant. One possible function could be:

$$f(x) = \frac{(x-1)(x-2)(x-3)x(x-4)}{x(x-4)}$$

### Concept Problem Revisited

To find the zeroes of a rational function, set the numerator equal to zero and solve for the  $x$  values. When a hole and a zero occur at the same point, the hole wins and there is no zero at that point.

### Vocabulary

A **zero** is where a function crosses the  $x$ -axis. It is also known as a root, solution or  $x$ -intercept.

A **rational function** is a function with at least one rational expression.

A **rational expression** is a ratio of two polynomial expressions.

### Guided Practice

1. Create a function with holes instead of zeroes.
2. Identify the  $y$  intercepts, holes, and zeroes of the following rational function.



$$f(x) = \frac{6x^3 - 7x^2 - x + 2}{x - 1}$$

3. Identify the zeroes and holes of the following rational function.

$$f(x) = \frac{2(x+1)(x+1)(x+1)}{2(x+1)}$$

**Answers:**

1. There are an infinite number of functions that meet the requirements. An illustrative example would be:

$$f(x) = (x - 1)(x + 1) \cdot \frac{(x-1)(x+1)}{(x-1)(x+1)}$$

The two  $x$  values that are holes are identical to the two  $x$  values that would be zeroes. Therefore, this function has no zeroes because holes exist in their place.

2. After noticing that a possible hole occurs at  $x = 1$  and using polynomial long division on the numerator you should get:

$$f(x) = (6x^2 - x - 2) \cdot \frac{x-1}{x-1}$$

A hole occurs at  $x = 1$  which turns out to be the point  $(1, 3)$  because  $6 \cdot 1^2 - 1 - 2 = 3$ .

The  $y$ -intercept always occurs where  $x = 0$  which turns out to be the point  $(0, -2)$  because  $f(0) = -2$ .

To find the  $x$ -intercepts you need to factor the remaining part of the function:

$$(2x + 1)(3x - 2)$$

Thus the zeroes ( $x$ -intercepts) are  $x = -\frac{1}{2}, \frac{2}{3}$ .

3. The hole occurs at  $x = -1$  which turns out to be a double zero. The hole still wins so the point  $(-1, 0)$  is a hole. There are no zeroes. The constant 2 in front of the numerator and the denominator serves to illustrate the fact that constant scalars do not impact the  $x$  values of either the zeroes or holes of a function.

## Practice

Identify the intercepts and holes of each of the following rational functions.

1.  $f(x) = \frac{x^3 + x^2 - 10x + 8}{x - 2}$

2.  $g(x) = \frac{6x^3 - 17x^2 - 5x + 6}{x - 3}$

3.  $h(x) = \frac{(x+2)(1-x)}{x-1}$

4.  $j(x) = \frac{(x-4)(x+2)(x+2)}{x+2}$

5.  $k(x) = \frac{x(x-3)(x-4)(x+4)(x+4)(x+2)}{(x-3)(x+4)}$

6.  $f(x) = \frac{x(x+1)(x+1)(x-1)}{(x-1)(x+1)}$

7.  $g(x) = \frac{x^3 - x^2 - x + 1}{x^2 - 1}$

8.  $h(x) = \frac{4 - x^2}{x - 2}$

9. Create a function with holes at  $x = 3, 5, 9$  and zeroes at  $x = 1, 2$ .

10. Create a function with holes at  $x = -1, 4$  and zeroes at  $x = 1$ .

11. Create a function with holes at  $x = 0, 5$  and zeroes at  $x = 2, 3$ .

12. Create a function with holes at  $x = -3, 5$  and zeroes at  $x = 4$ .

13. Create a function with holes at  $x = -2, 6$  and zeroes at  $x = 0, 3$ .

14. Create a function with holes at  $x = 1, 5$  and zeroes at  $x = 0, 6$ .

15. Create a function with holes at  $x = 2, 7$  and zeroes at  $x = 3$ .

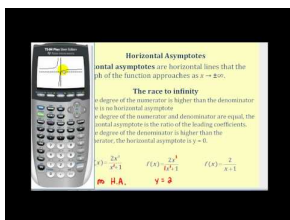
## 3.5 Vertical Asymptotes

Here you will learn to recognize when vertical asymptotes occur and what makes them different from removable discontinuities.

The basic rational function  $f(x) = \frac{1}{x}$  is a hyperbola with a vertical asymptote at  $x = 0$ . More complicated rational functions may have multiple vertical asymptotes. These asymptotes are very important characteristics of the function just like holes. Both holes and vertical asymptotes occur at  $x$  values that make the denominator of the function zero. A driving question is: what makes vertical asymptotes different from holes?

### Watch This

Watch the following video, focusing on the parts about vertical asymptotes.



### MEDIA

Click image to the left or use the URL below.

URL: <http://www.ck12.org/flx/render/embeddedobject/60833>

<http://www.youtube.com/watch?v=wBZxVxiJS9I> James Sousa: Determining Horizontal and Vertical Asymptotes of Rational Functions

### Guidance

Vertical asymptotes occur when a factor of the denominator of a rational expression does not cancel with a factor from the numerator. When you have a factor that does not cancel, instead of making a hole at that  $x$  value, there exists a vertical asymptote. The vertical asymptote is represented by a dotted vertical line. Most calculators will not identify vertical asymptotes and some will incorrectly draw a steep line as part of a function where the asymptote actually exists.

Your job is to be able to identify vertical asymptotes from a function and describe each asymptote using the equation of a vertical line.

### Example A

Identify the holes and the equations of the vertical asymptotes for the following rational function.

$$f(x) = \frac{(2x-3)(x+1)(x-2)}{(x+2)(x+1)}$$

**Solution:** The factor that cancels represents the removable discontinuity. There is a hole at  $(-1, 15)$ . The vertical asymptote occurs at  $x = -2$  because the factor  $x + 2$  does not cancel.

### Example B

Identify the domain of the following function and then identify the holes and vertical asymptotes.

$$f(x) = \frac{(3x-4)(1-x)(x^2+4)}{(3x-2)(x-1)}$$

**Solution:** The domain of the function written in interval notation is:  $(-\infty, \frac{2}{3}) \cup (\frac{2}{3}, 1) \cup (1, \infty)$

There are two discontinuities: one is a hole and one is a vertical asymptote. The hole occurs at  $(1, 5)$ . The vertical asymptote occurs at  $x = \frac{2}{3}$ .

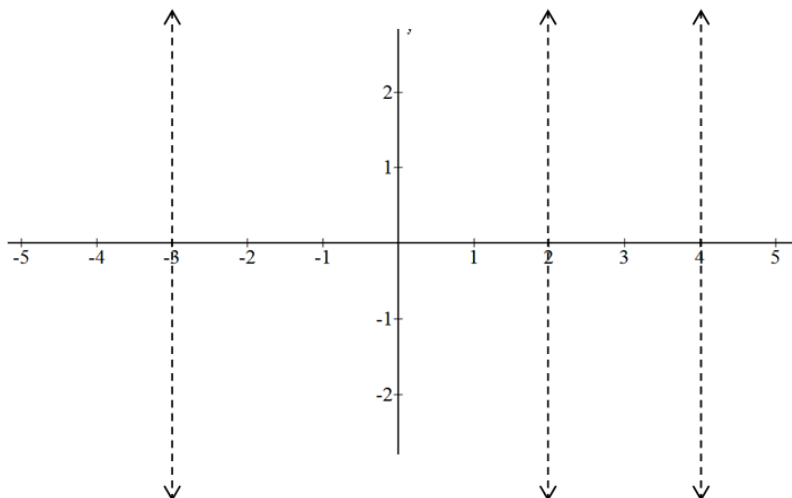
Notice that holes are identified as points while vertical asymptotes are identified as lines of the form  $x = a$  where  $a$  is some constant.

### Example C

Draw the vertical asymptotes for the following function.

$$f(x) = \frac{1}{(x-4)(x-2)(x+3)}$$

**Solution:**



Note that you may not know the characteristics of what the function does inside these vertical lines. You will soon learn how to use sign tests as well as techniques you've already learned to fill in the four sections that this function is divided into.

### Concept Problem Revisited

Holes occur when factors from the numerator and the denominator cancel. When a factor in the denominator does not cancel, it produces a vertical asymptote. Both holes and vertical asymptotes restrict the domain of a rational function.

### Vocabulary

A **vertical asymptote** is a dashed vertical line that indicates that as a function approaches, it shoots off to positive or negative infinity without ever actually touching the line.

A **rational function** is a function with at least one rational expression.

A **rational expression** is a ratio of two polynomial expressions.

### Guided Practice

1. Write a function that fits the following criteria:

- Vertical asymptotes at 0 and 3
- Zeros at 2 and 5
- Hole at  $(4, 2)$

2. Draw the vertical asymptotes for the following function.

$$f(x) = 3 - \frac{x}{(x+1)(x-4)}$$

3. Identify the holes and equations of the vertical asymptotes of the following rational function.

$$f(x) = \frac{3(x-1)(x+2)(x-3)(x+4)}{5\left(x+\frac{1}{2}\right)(2+x)(3-x)(x-8)}$$

### Answers:

1. Each criteria helps build the function. The vertical asymptotes imply that the denominator has two factors that do not cancel with the numerator:

$$\frac{1}{x \cdot (x-3)}$$

The zeroes at 2 and 5 imply the numerator has two factors that do not cancel.

$$\frac{(x-2)(x-5)}{x \cdot (x-3)}$$

The hole at (4, 2) implies that there is a factor  $x - 4$  that cancels on the numerator and the denominator.

$$\frac{(x-2)(x-5)(x-4)}{x \cdot (x-3)(x-4)}$$

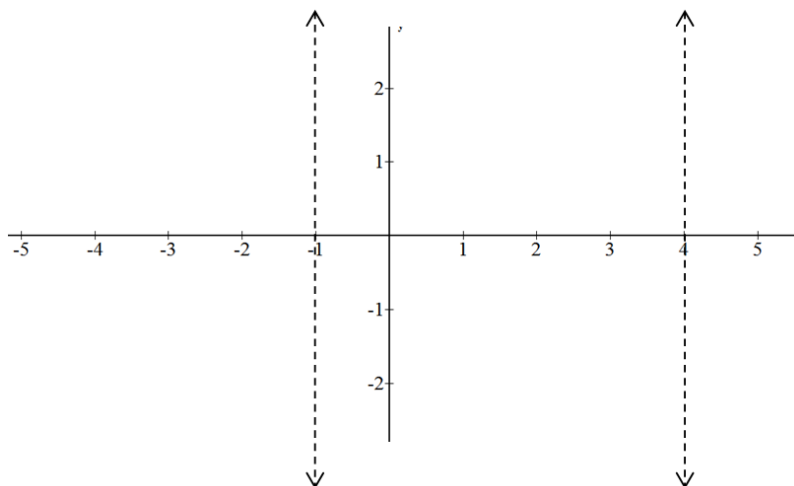
The tricky part is that the height of the function must be 2 after the  $x - 4$  factor has been canceled and the 4 is substituted in. Currently it is  $-\frac{1}{2}$ .

$$\frac{(4-2)(4-5)}{4 \cdot (4-3)} = -\frac{1}{2}$$

In order to make the hole exist at a height of 2, you need to multiply the function by a scalar of -4.

$$f(x) = \frac{-4(x-2)(x-5)(x-4)}{x \cdot (x-3)(x-4)}$$

2. The vertical asymptotes occur at  $x = -1, x = 4$ .



3. The vertical asymptotes occur at  $x = -\frac{1}{2}, x = 8$ . Holes occur when  $x$  is -2 and 3. To get the height of the holes at these points, remember to cancel what can be canceled and then substitute the values. A very common mistake is to forget to cancel  $\frac{x-3}{3-x} = -1$ .

$$g(x) = \frac{-3(x-1)(x+4)}{5\left(x+\frac{1}{2}\right)(x-8)}$$

$$g(-2) = \frac{6}{25}$$

$$g(3) = \frac{12}{25}$$

The holes are at  $(-2, \frac{6}{25})$ ,  $(3, \frac{12}{25})$ .

### Practice

1. Write a function that fits the following criteria:

- Vertical asymptotes at 1 and 4
- Zeroes at 3 and 5
- Hole at (6, 3)

2. Write a function that fits the following criteria:

- Vertical asymptotes at -2 and 2
- Zeroes at 1 and 5
- Hole at (3, -4)

3. Write a function that fits the following criteria:

- Vertical asymptotes at 0 and 3
- Zeroes at 1 and 2
- Hole at (8, 21)

4. Write a function that fits the following criteria:

- Vertical asymptotes at 2 and 6
- Zero at 5
- Hole at (4, 1)

5. Write a function that fits the following criteria:

- Vertical asymptote at 4
- Zeroes at 0 and 3
- Hole at (5, 10)

Give the equations of the vertical asymptotes for the following functions.

6.  $f(x) = \frac{(2-x)}{(x-2)(x-4)}$

7.  $g(x) = \frac{-x}{(x+1)(x-3)}$

8.  $h(x) = 6 - \frac{x+2}{(x+1)(x-5)}$

9.  $j(x) = \frac{10}{x-3} - \frac{x}{(x+2)(x-3)}$

10.  $k(x) = 2 - \frac{(4-x)}{(x+3)(x-4)}$

Identify the holes and equations of the vertical asymptotes of the following rational functions.

11.  $f(x) = \frac{3(x-1)(x+1)(x-4)(x+4)}{4(x+4)(2+x)(4-x)(x+1)}$

12.  $g(x) = \frac{x(x-3)(x-8)(x-3)(x+4)}{7(x+1)(1+x)(3-x)(x-8)}$

State the domain of the following rational functions.

13.  $h(x) = \frac{x(x+1)(x-3)(x+4)}{x(3-x)(x-1)}$

$$14. j(x) = \frac{x^2+3x-4}{x^2-6x-16}$$

$$15. k(x) = \frac{2x-10}{x^3+4x^2+3x}$$

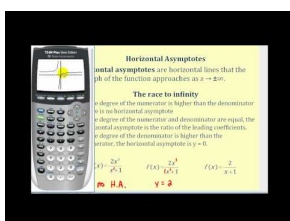
## 3.6 Horizontal Asymptotes

Here you will learn to identify when horizontal asymptotes exist, specify their height and write their equation.

Vertical asymptotes describe the behavior of a function as the values of  $x$  approach a specific number. Horizontal asymptotes describe the behavior of a function as the values of  $x$  become infinitely large and infinitely small. Since functions cannot touch vertical asymptotes, are they not allowed to touch horizontal asymptotes either?

### Watch This

Watch the following video, focusing on the parts about horizontal asymptotes.



### MEDIA

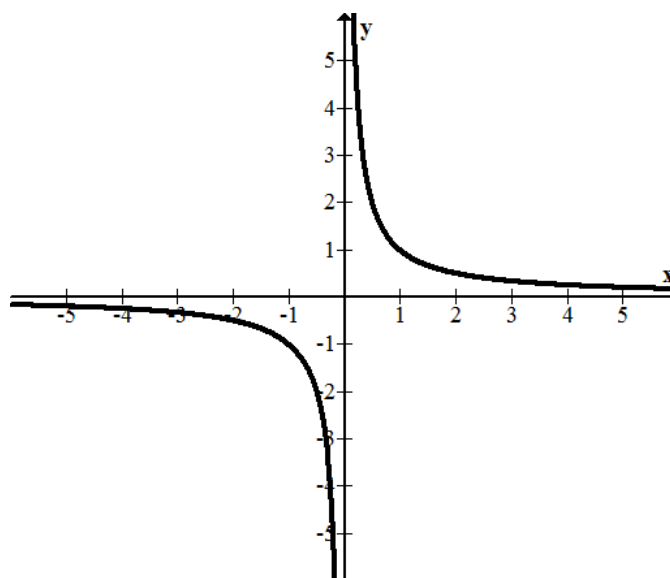
Click image to the left or use the URL below.

URL: <http://www.ck12.org/flx/render/embeddedobject/60833>

<http://www.youtube.com/watch?v=wBZxVxiJS9I> James Sousa: Determining Horizontal and Vertical Asymptotes of Rational Functions

### Guidance

**Horizontal asymptotes** are a means of describing end behavior of a function. End behavior essentially is a description of what happens on either side of the graph as the function continues to the right and left infinitely. When you are determining the horizontal asymptotes, it is important to consider both the right and the left hand sides, because the horizontal asymptotes will not necessarily be the same in both places. Consider the reciprocal function and note how as  $x$  goes to the right and left it flattens to the line  $y = 0$ .



Sometimes functions flatten out and other times functions increase or decrease without bound. There are basically three cases.

**Case 1:** The first case is the function flattens out to 0 as  $x$  gets infinitely large or infinitely small. This happens when the degree of the numerator is less than the degree of the denominator. The degree is determined by the greatest exponent of  $x$ .

$$f(x) = \frac{2x^8 + 3x^2 + 100}{x^9 - 12}$$

One way to reason through why this makes sense is because when  $x$  is a ridiculously large number then most parts of the function hardly make any impact. The 100 for example is nothing in comparison and neither is the  $3x^2$ . The two important terms to compare are  $x^8$  and  $x^9$ . The 2 isn't even important now because if  $x$  is even just a million then the  $x^9$  will be a million times bigger than the  $x^8$  and the 2 hardly matters again. Essentially, when  $x$  gets big enough, this function acts like  $\frac{1}{x}$  which has a horizontal asymptote of 0.

**Case 2:** The degree of the numerator is equal to the degree of the denominator. In this case, the horizontal asymptote is equal to the ratio of the leading coefficients.

$$f(x) = \frac{6x^4 - 3x^3 + 12x^2 - 9}{3x^4 + 144x - 0.001}$$

Notice how the degree of both the numerator and the denominator is 4. This means that the horizontal asymptote is  $y = \frac{6}{3} = 2$ . One way to reason through why this makes sense is because when  $x$  gets to be a very large number all the smaller powers will not really make much of an impact. The biggest contributors are only the biggest powers. Then the value of the numerator will be about twice that of the denominator. As  $x$  gets even bigger, then the function will get even closer to 2.

**Case 3:** The degree of the numerator is greater than the degree of the denominator. In this case there does not exist a horizontal asymptote and you must determine if the function increases or decreases without bound in both the left and right directions.

### Example A

Identify the horizontal asymptotes of the following 3 functions:

- $f(x) = \frac{4x^3 + 99}{3x^4 - 99}$
- $h(x) = \frac{234x^{45} - 45x^{234} + 100}{x^{235}}$
- $g(x) = \frac{x^3 + 3x^6}{x^3 - 6x^6}$

### Solution:

- $y = 0$  because the degree of the numerator is 3 and the degree of the denominator is 4, so the denominator gets bigger eventually and the fraction approaches 0.
- $y = 0$  because the degree of the numerator is 234 which is smaller than the degree of the denominator (235).
- $y = -\frac{1}{2}$  because the degree of both the numerator and the denominator is 6 so the horizontal asymptote is the ratio of the leading coefficients. Note that leading refers to the coefficients of the greatest powers of  $x$  not the coefficients that happen to be written out front. Convention usually tells you to write the powers of  $x$  in descending order.



**Example B**

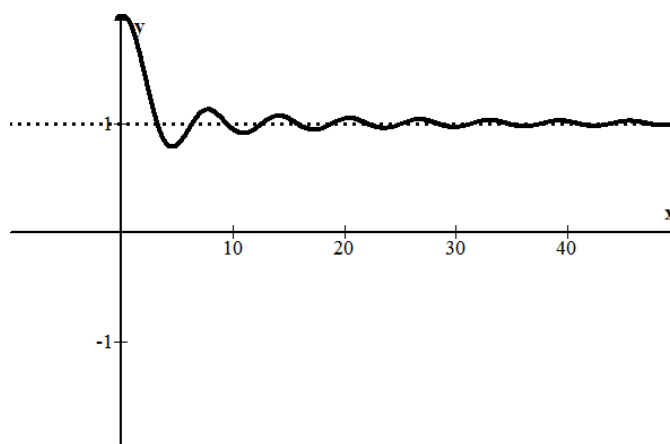
Identify the vertical and horizontal asymptotes of the following rational function.

$$f(x) = \frac{(x-2)(4x+3)(x-4)}{(x-1)(4x+3)(x-6)}$$

**Solution:** There is factor that cancels that is neither a horizontal or vertical asymptote. The vertical asymptotes occur at  $x = 1$  and  $x = 6$ . To obtain the horizontal asymptote you could methodically multiply out each binomial, however since most of those terms do not matter, it is more efficient to first determine the relative powers of the numerator and the denominator. In this case they both happen to be 3. Next determine the coefficient of the cubic terms only. The numerator will have  $4x^3$  and the denominator will have  $4x^3$  and so the horizontal asymptote will occur at  $y = \frac{4}{4} = 1$ .

**Example C**

Describe the right hand end behavior of the following function.



**Solution:** Notice how quickly this dampening wave function settles down. There seems to be an obvious horizontal axis on the right at  $y = 1$

**Concept Problem Revisited**

As in Example C, functions may touch and pass through horizontal asymptotes without limit. This is a difference between vertical and horizontal asymptotes. In calculus, there are rigorous proofs to show that functions like the one in Example C do become arbitrarily close to the asymptote.

**Vocabulary**

A **horizontal asymptote** is a flat dotted line that indicates where a function goes as  $x$  get infinitely large or infinitely small.

**End behavior** is a term that asks you to describe the horizontal asymptotes.

A **vertical asymptote** is a dashed vertical line that indicates that as a function approaches, it shoots off to positive or negative infinity without ever actually touching the line.

A **rational function** is a function with at least one rational expression.

A **rational expression** is a ratio of two polynomial expressions.

**Guided Practice**

1. Identify the horizontal asymptotes of the following function.

$$f(x) = \frac{(x-3)(x+2)}{|(x-5)| \cdot (x-1)}$$

2. Identify the vertical and horizontal asymptotes and the holes of the following function.

$$f(x) = \frac{(x^4 - 9)(x - 1)}{(x^2 + 3)(x - 3)}$$

3. Identify the horizontal asymptotes if they exist for the following 3 functions.

a.  $f(x) = \frac{3x^6 - 72x}{x^6 + 999}$

b.  $h(x) = \frac{ax^4 + bx^3 + cx^2 + dx + e}{fx^4 + gx^3 + hx^2}$

c.  $g(x) = \frac{f(x)}{h(x)}$

**Answers:**

1. First notice the absolute value surrounding one of the terms in the denominator. The degrees of both the numerator and the denominator will be 2 which means that the horizontal asymptote will occur at a number. As  $x$  gets infinitely large, the function is approximately:

$$f(x) = \frac{x^2}{x^2}$$

So the horizontal asymptote is  $y = 1$  as  $x$  gets infinitely large.

On the other hand, as  $x$  gets infinitely small the function is approximately:

$$f(x) = \frac{x^2}{-x^2}$$

So the horizontal asymptote is  $y = -1$  as  $x$  gets infinitely small.

In this case, you cannot blindly use the leading coefficient rule because the absolute value changes the sign.

2. The numerator of the function factors to be:

$$f(x) = \frac{(x^2 + 3)(x^2 - 3)(x - 1)}{(x^2 + 3)(x - 3)} = \frac{(x^2 - 3)(x - 1)}{x - 3}$$

Note that a factor does cancel and also notice that this factor is never equal to zero. Not all factors that cancel indicate a hole. A horizontal asymptote does not exist because the degree of the numerator is greater than the degree of the denominator. The vertical asymptote is at  $x = 3$ .

3.

- The degrees of the numerator and the denominator are equal so the horizontal asymptote is  $y = 3$ .
- The degrees of the numerator and the denominators are equal again so the horizontal asymptote is  $y = \frac{a}{f}$ .
- As  $x$  gets infinitely large,

$$g(x) = \frac{f(x)}{h(x)} = \frac{\frac{3x^6 - 72x}{x^6 + 999}}{\frac{ax^4 + bx^3 + cx^2 + dx + e}{fx^4 + gx^3 + hx^2}} \approx \frac{3}{\frac{a}{f}} = \frac{3f}{a}$$

When you study calculus, you will learn the rigorous techniques that enable you to feel more confident about results like this.

## Practice

Identify the horizontal asymptotes, if they exist, for the following functions.

- $f(x) = \frac{5x^4 - 2x}{x^4 + 32}$
- $g(x) = \frac{3x^4 - 2x^6}{-x^4 + 2}$
- $h(x) = \frac{3x^4 - 5x}{8x^3 + 3x^4}$
- $j(x) = \frac{2x^3 - 15x}{-x^4 + 3}$
- $k(x) = \frac{2x^5 - 3x}{5x^2 + 3x^4 + 2x - 7x^5}$
- $f(x) = \frac{ax^{14} + bx^{23} + cx^{12} + dx + e}{fx^{24} + gx^{23} + hx^{21}}$
- $g(x) = \frac{(x-1)(x+4)}{|(x-2)| \cdot (x-1)}$
- Write a function that fits the following criteria:
  - Vertical asymptotes at  $x = 1$  and  $x = 4$
  - Zeros at 3 and 5
  - Hole when  $x = 6$
  - Horizontal asymptote at  $y = \frac{2}{3}$
- Write a function that fits the following criteria:
  - Vertical asymptotes at  $x = -2$  and  $x = 2$
  - Zeros at 1 and 5
  - Hole when  $x = 3$
  - Horizontal asymptote at  $y = 1$
- Write a function that fits the following criteria:
  - Vertical asymptotes at  $x = 0$  and  $x = 3$
  - Zeros at 1 and 2
  - Hole when  $x = 8$
  - Horizontal asymptote at  $y = 2$
- Write a function that fits the following criteria:
  - Vertical asymptotes at 2 and 6
  - Zero at 5
  - Hole when  $x = 4$

- Horizontal asymptote at  $y = 0$

12. Write a function that fits the following criteria:

- Vertical asymptote at 4
- Zeroes at 0 and 3
- Hole at when  $x = 5$
- No horizontal asymptotes

Identify the vertical and horizontal asymptotes of the following rational functions.

$$13. f(x) = \frac{(x-5)(2x+1)(x-3)}{(x-3)(4x+5)(x-6)}$$

$$14. g(x) = \frac{x(x-1)(x+3)(x-5)}{3x(x-1)(4x+3)}$$

$$15. h(x) = \frac{(x-2)(x+3)(x-6)}{(x-4)(x+3)^2(x+2)}$$

Polynomials and rational functions were explored. Special features of rational functions such as holes and asymptotes were introduced. Graphing rational functions by hand, which utilized work from many concepts, concluded the chapter.

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## 3.7 References

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## CHAPTER

## 4

# Exponentials

## Chapter Outline

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- 4.1      EXPONENTIAL FUNCTIONS AND THE 2 INTEREST FORMULAS**
  - 4.2      THE MORTGAGE FORMULA**
  - 4.3      EXPONENTIAL MODELING AND THE LAW OF UNINHIBITED GROWTH**
- 

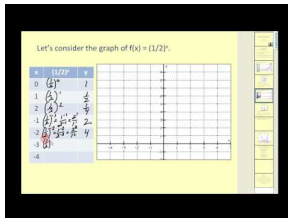
Exponential growth has been called one of the most powerful forces in the universe. You may already know the basic rules of exponents. Here you will explore their relationship with logarithms and their application to different types of growth and decay.

## 4.1 Exponential Functions and The 2 Interest Formulas

Here you will explore exponential functions as a way to model a special kind of growth or decay and you will learn more about the number  $e$ .

Exponential growth is one of the most powerful forces in nature. A famous legend states that the inventor of chess was asked to state his own reward from the king. The man asked for a single grain of rice for the first square of the chessboard, two grains of rice for the second square and four grains of rice for the third. He asked for the entire 64 squares to be filled in this way and that would be his reward. Did the man ask for too little, or too much?

### Watch This



### MEDIA

Click image to the left or use the URL below.

URL: <http://www.ck12.org/flx/render/embeddedobject/60994>

<http://www.youtube.com/watch?v=7fpazNs1ZRE> James Sousa: Graph Exponential Functions

### Exponential and Logarithmic Functions

#### Do Now

Round on your calculator to the nearest 4 decimal places :

1)  $5000 (2^{-1.5})$       3)  $^3\sqrt{4395}$

2)  $(1.005)^{400}$       4)  $4^{pi}$

1) 1,767.7670

2) 7.3523

3) 16.3802

4) 77.8802

There is an irrational number in math called the number “e”

$$e = 2.7281828....$$

“e” has many uses and applications such as money problems, and chemistry applications such as half-life and radioactive decay.

“e” has its own button on the calculator

2nd ln

2nd divided by

Try these - round to three decimal places

1)  $e^3$

**Answer: 20.086**

2)  $-2e^{-5}$

**Answer: -0.013**

3)  $2 + e$

**Answer: 4.718**

## MONEY PROBLEMS

### TWO INTEREST FORMULAS

There are two different interest formulas we will be using:

#### Interest Formula # 1

$$A = P ( 1 + r/n )^{nt}$$

A = Balance (\$ in the account after the time has gone by)

P = Principal (\$ invested)

r = Interest rate (as a decimal)

n = Number of compoundings per year



**t = time in years the \$ is invested for**

**Number of compoundings per year**

**quarterly n = 4**

**semiannually n = 2**

**monthly n = 12**

**weekly n = 52**

**daily n = 365**

**ex) \$ 10,000 is invested at 8% for 5 years. If the interest is compounded quarterly, what will be the balance ?**

$$A = P ( 1 + r/n )^{nt}$$

**First determine your variables**

**P =**

**r =**

**n =**

**t =**

**ex) \$ 10,000 is invested at 8% for 5 years. If the interest is compounded quarterly, what will be the balance ?**

$$A = P ( 1 + r/n )^{nt}$$

**P = 10,000**

**r = 0.08 (8 % as a decimal)**

**n = 4 (quarterly)**

**t = 5**

$$A = 10,000 ( 1 + 0.08/4 )^{(4)(5)}$$

**A = \$ 14,859.47**

**(always round to the nearest cent)**

### YOU TRY

**You deposit \$ 5,000 in a trust fund that pays 7.5 % interest for 50 years. What is the balance if the interest is compounded semiannually?**

semiannually

$$A = P ( 1 + r/n )^{nt}$$

$$P = 5000$$

$$r = .075$$

$$n = 2$$

$$t = 50$$

$$A = 5000 ( 1 + 0.075/2 )^{(2)(50)}$$

$$\text{\$ } 198,509.16$$

**Here is a video to assist you with Interest Formula 1** Interest Formula 1 <http://www.educations.com/lesson/view/interest-formula-1/18358604/?ref=app>

### Interest Formula # 2

**For continuous compounding of interest**

$$A = Pe^{rt}$$

**A = Balance**

**P = Principal (\$ invested)**

**r = interest rate (as decimal)**

**t = time in years**

**e is just e (2.71828...)**

**Ex) \$ 12,000 is invested in an account which pays 7.75 % interest compounded continuously. What would the balance of the account be after 5 years ?**

$$A = Pe^{rt}$$

$$P = 12,000$$

$$r = .0775$$

$$t = 5$$

$$A = Pe^{rt}$$

$$A = 12,000e^{(.0775)(5)}$$

**ANSWER: \$ 17,679.52**

Here is a video to assist you with Interest Formula 2

**Interest formula 2** <http://www.educations.com/lesson/view/interest-formula-2/18715795/?ref=app>

Ex) Suppose you deposit

\$ 5,000 in a trust fund that pays 7.5 % interest for 50 years. Which of the following interest options yields the greatest balance?

a) monthly

b) continuously

a) monthly

$$A = P ( 1 + r/n )^{nt}$$

**P = 5000**

$$r = .075$$

$$n = 12$$

$$t = 50$$

$$A = 5000 ( 1 + 0.075/12 )^{(12)(50)}$$

**Answer: \$ 210,138.69**

b) continuously

$$A = Pe^{rt}$$

$$P = 5000$$

$$R = .075$$

$$T = 50$$

$$A = 5000e^{(.075)(50)}$$

**Answer: \$ 212,605.41**

## 4.2 The Mortgage Formula

Here you will learn how exponents interact in a variety of algebraic situations including addition, subtraction, multiplication and exponentiation.

A mortgage loan, also referred to as a mortgage, is used by purchasers of real estate when they do not possess enough money to purchase the home initially. To calculate the monthly payment for a mortgage payment, you use the following formula:

$$\text{Payment} = \frac{M (1 + r/12)^{12Y} \times r/12}{(1 + r/12)^{12Y} - 1}$$

The Mortgage Formula <http://www.educreations.com/lesson/view/mortgage-formula/19051558/?ref=app>

In order to purchase a home, a bank requires a downpayment to be paid to the bank. In addition, included in a mortgage payment are the taxes that are owed on the home. Heres a video to explain the overall home purchase procedure: Complete Mortgage Problems <http://www.educreations.com/lesson/view/full-mortgage-problems/19148463/?ref=app>

Here is how you store the mortgage formula into the program section of the graphing calculator: Storing the Mortgage Formula in your Programs <http://www.educreations.com/lesson/view/storing-mortgage-formula-to-the-programs-of-your-c/19148462/?ref=app>

tgage

## 4.3 Exponential Modeling and The Law of Uninhibited Growth

Here you will apply the new algebraic techniques associated with logs to solve equations.

1) How long does it take for an investment of \$ 5,000 double if the interest is compounded continuously at 7.5 % ?

Continuously compounded interest

$$A = Pe^{rt}$$

$$P = 5,000$$

$$R = 0.075$$

$$T = ?$$

$A = 10,000$  b/c the investment is doubling

Plug in and solve for t:

$$A = Pe^{rt}$$

$$10,000 = 5,000e^{0.075t}$$

Enter the left side of the equation (10,000) into Y1 of your calculator.

Enter the right side of the equation ( $5,000e^{0.075t}$ ) into Y2 of your calculator

Select a viewing window to calculate the point of intersection of Y1 and Y2.

$$XMin = ?$$

$$XMax = ?$$

$$YMin = ?$$

$$YMax = ?$$

You are solving for x. You are trying to calculate how long it will take for the money to double so assume it will occur sometime in the next 100 years so:

$$XMin = 0$$

$$XMax = 100$$

Since you are trying to solve the equation  $10,000 = 5,000e^{0.075t}$

$$Ymin = 0$$

$$Ymax = 15,000 \text{ (since the biggest number in the equation is 10,000)}$$

When you find the point of intersect, the x value is the answer:

$$t = 9.241962407$$

$$t = 9.2 \text{ years}$$

### **EXPONENTIAL MODELING - THE LAW OF UNINHIBITED GROWTH**

The following is the formula for the Law of Uninhibited Growth:

$$y = Ce^{kt}$$

where

C = original value / amount when  $t = 0$

k = exponential variable

- the value changes every problem

- always round to 4 decimals

t = time in years

y = value / amount after t years.

ex) A car that costs \$ 45,000 new has a depreciated value of \$ 32,000 after 3 years. Find the value of the car after 5 years using the Law of Uninhibited Growth  $y = Ce^{kt}$ .

$$y = Ce^{kt}$$

$$y = 32,000 \text{ (\$ after } t \text{ years)}$$

$$C = 45,000 \text{ (initial value when } t = 0)$$

k = you are always going to find this value first

$$t = 3$$

$$y = Ce^{kt}$$

$$32,000 = 45,000e^{3k}$$

Solve for k.

Enter the left side of the equation (32,000) into Y1 of your calculator.

Enter the right side of the equation ( $45,000e^{3k}$ ) into Y2 of your calculator

Select a viewing window to calculate the point of intersection of Y1 and Y2.

Note: k is always a decimal between -1 and +1.

If the values in the problem are iNCREASING, k is positive.

If the values in the problem are DECREASING, k is negative.

So always set the:

$$XMin = -2$$

$$XMax = +2$$

$$32,000 = 45,000e^{3k}$$

$$Y_{\text{Min}} = 0$$

To get the  $Y_{\text{Max}}$ , just make the values bigger than the biggest number that you see  $Y_{\text{Max}} = 50,000$ . Find the point of intersection of  $Y_1$  and  $Y_2$  and the x-value of the point of intersection is the  $k$  value. Always round to 4 decimal places.

$$k = -0.1136$$

Now plug in  $k$  and  $C$  and solve for  $y$ :

A car that costs \$ 45,000 new has a depreciated value of \$ 32,000 after 3 years. Find the value of the car after 5 years using the exponential equation  $y = Ce^{kt}$

$$y = 45,000e^{-0.1136t}$$

Plug in  $n = 5$ .

$$y = 45,000e^{-0.1136t}$$

$$y = 45,000e^{-0.1136(5)}$$

$$y = 45,000e^{-0.568}$$

$$y = \$ 25,499.59$$

Exponential functions demonstrate applications of geometric growth and decay in the real world. After practicing with the rules and procedures to gain fluency with exponents and scientific notation, you transferred your knowledge to logarithms and their properties. Lastly, you explored how a new type of function, the logistic function, improves on exponential growth models for real world application.

## CHAPTER

## 5

# Inequalities

## Chapter Outline

---

**5.1 THE INEQUALZ APP ON THE GRAPHING CALCULATOR**

**5.2 LINEAR PROGRAMMING**

**5.3 INEQUALITIES AND ECONOMICS**

---

qHere you will review concepts of exponential growth and geometric series with a focus on the relationship between time and money.



## 5.1 The Inequalz App on the Graphing Calculator

Here you'll learn to calculate the effect of time on the balance of a savings account growing by simple interest.

### GRAPHING INEQUALITIES ON THE CALCULATOR

**Graph:**  $y \geq 3$  on your calculator.

To do so use the APPS ALPHA  $x^2$  (I)

Inequalz

Press any key

Now on the Y= screen, anytime the cursor is over the =, you have the option of inserting an inequality symbol instead of =.

**TO CHANGE FROM = TO an Inequality Symbol**

-

ALPHA   =   <   ≤   >   ≥

Y=   WINDOW   ZOOM   TRACE   GRAPH

So to graph  $y \geq 3$

Move cursor to = and press ALPHA GRAPH and then type in 3.

Graph in the standard viewing window

You are going to be asked questions like

“Name a point in the solution area and verify algebraically”

**TO SHUT OFF INEQUALZ****APPS      ALPHA X<sup>2</sup> (I)****1:Continue****2:Quit Inequal      ß Select option 2****3: About**

**View this video for a more detailed explanation of the INEQUALZ app of the graphing calculator:**

**Using the Inequalz app** <https://www.educations.com/lesson/view/using-the-inequalz-app/26668953/?s=IIMWFJ&ref=app>

***Principal*** is the amount initially deposited into the account. *Notice the spelling is principal, not principle.* ***Interest*** is the conversion of time into money.

## 5.2 Linear Programming

Here you'll explore how to compute an investment's growth given time and a compound interest rate.

Linear programming (also called linear optimization) is a method to achieve the best outcome (such as maximum profit or lowest cost) in a [mathematical model](#) whose requirements are represented by linear relationships. Linear programming is a special case of mathematical programming ([mathematical optimization](#)).

More formally, linear programming is a technique for the [optimization](#) of a [linear objective function](#), subject to [linear equality](#) and [linear inequality constraints](#).

Linear programming can be applied to various fields of study. It is used in business and [economics](#), but can also be utilized for some engineering problems. Industries that use linear programming models include transportation, energy, telecommunications, and manufacturing. It has proved useful in modeling diverse types of problems in planning, [routing](#), [scheduling](#), [assignment](#), and design.

Here is an example of what Linear Programming "looks like" and what the procedure is going to be:

### LINEAR PROGRAMMING

#### Objective Function:

$M = 3x + 5y \rightarrow$  The equation to be maximized or minimized

#### Constraints:

$x \geq 0$	Graph all the constraints on the same graph and find the vertices of the solution area.
$y \geq 0$	
$x + y \leq 6$	
$3x - y \leq 4$	

Then plug the vertices of the solution area into the objective function and answer the question.

View this video for a detailed explanation of the procedure for Linear Programming and the "Solution" for the Above Problem:

LINEAR PROGRAMMING - procedure <https://www.educations.com/lesson/view/linear-programming-procedure>

[e/27105170/?ref=app](https://www.ck12.org/e/27105170/?ref=app)

### Steps for Linear Programming Problems

- 1) Write the 2 let statements.
- 2) Organize the information by using a chart if possible.
- 3) Identify the objective function to be maximized or minimized. (The row without the total)
- 4) Write the constraints (inequalities) and don't forget the minimum value inequalities.
- 5) Graph the inequalities and find the vertices of the solution area.
- 6) Plug the vertices into the objective function and answer the question. (Remember the point is the answer)

Here is a video detailing a Linear Programming Word Problem: **Linear Programming Model Problem** <https://www.educations.com/lesson/view/linear-programming-worksheet-2/14484353/?ref=app>

L

## 5.3 Inequalities and Economics

<https://www.educreations.com/lesson/view/inequalities-and-economics/13983698/?ref=link> Consumer surplus

Consumer surplus is derived whenever the price a consumer actually pays is less than they are prepared to pay. A demand curve indicates what price consumers are prepared to pay for a hypothetical quantity of a good, based on their expectation of private benefit.

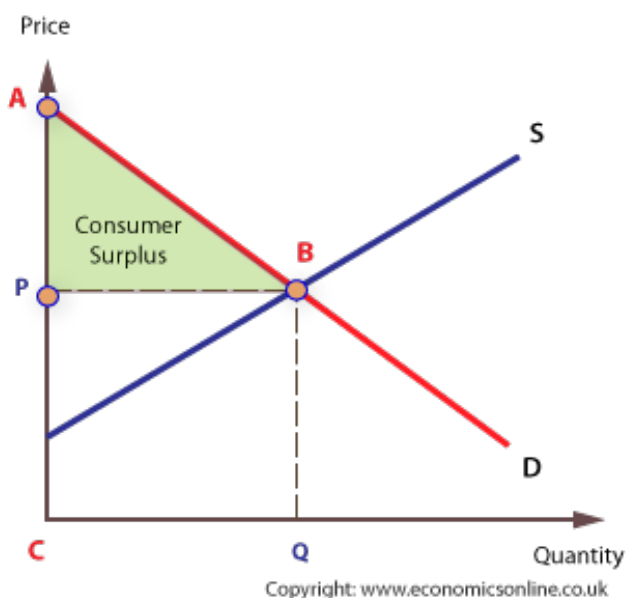


FIGURE 5.1

The amount consumers actually spend is determined by the market price they pay,  $P$ , and the quantity they buy,  $Q$  - namely,  $P \times Q$ , or area  $PBQC$ . This means that there is a net gain to the consumer, because area  $ABQC$  is greater than area  $PBQC$ . This net gain is called *consumer surplus*, which is the total benefit, area  $ABQC$ , less the amount spent, area  $PBQC$ . Hence  $ABQC - PBQC = \text{area } ABP$ . **Producer surplus**

Producer surplus is the additional private benefit to producers, in terms of *profit*, gained when the price they receive in the market is more than the minimum they would be prepared to supply for. In other words they received a reward that more than covers their *costs* of production.

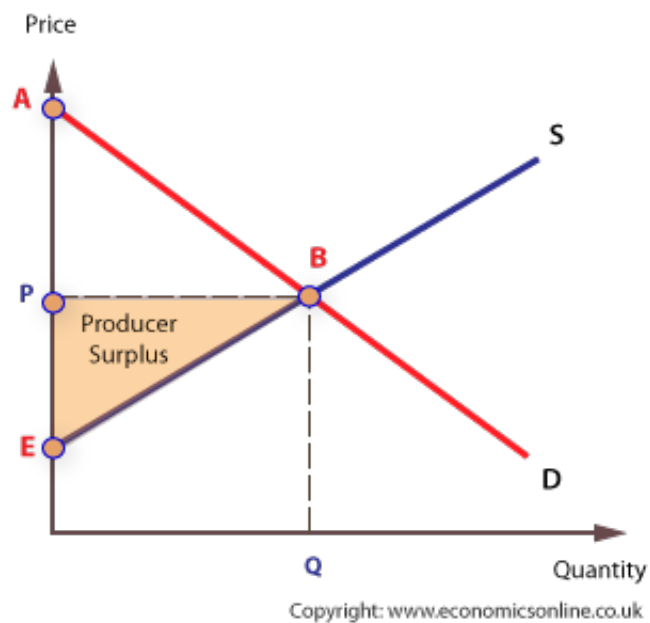


FIGURE 5.2

The effects of interest on lump sum deposits and periodic deposits were explored. The key idea was that a dollar today is worth more than a dollar in a year.

## CHAPTER

## 6

# Concepts of Statistics

## Chapter Outline

---

- 6.1 STATISTICS ON THE GRAPHING CALCULATOR - MEAN, MEDIAN AND MODE**
  - 6.2 STATISTICAL DEVIATION**
  - 6.3 FIVE NUMBER SUMMARY, BOX AND WHISKER PLOTS, AND MODIFIED BOX PLOTS AND OUTLIERS**
  - 6.4 GRAPHIC DISPLAYS OF DATA - BAR GRAPHS, PIE CHARTS, HISTOGRAMS, CUMULATIVE HISTOGRAMS, STEM AND LEAF**
  - 6.5 OTHER TYPES OF AVERAGES**
  - 6.6 PERCENTILE RANK**
  - 6.7 LINEAR CORRELATION**
  - 6.8 OTHER REGRESSION TYPES REGRESSION**
  - 6.9 REFERENCES**
- 

Statistics is hugely important for understanding, describing and predicting the world around you. Descriptive statistics is using summaries to present information that you have found to a reader. Summaries can be graphs or small groups of numbers that are easier to understand than long lists of numbers. Inferential statistics is using data to make predictions. Both inferential statistics and descriptive statistics help you understand the world around you and communicate it effectively.

## 6.1 Statistics on the Graphing Calculator - Mean, Median and Mode

Here you will calculate three measures of the center of univariate data and decide which measure is best based on context. **This chapter will cover statistics and the use of the STAT mode of a graphing calculator. Please view this video which details how to enter data into a graphing calculator.**

**Statistics on the Graphing Calculator** <http://www.educreations.com/lesson/view/statistics-on-the-ti-84-plus-silver/15326242/?ref=appemail> **Oftentimes, you are going to want to organize data in either ascending or descending order. Please view this video which covers those aspects of the STAT mode of a graphing calculator.**

**Sort and Frequency Distribution Features on the Calculator** <http://www.educreations.com/lesson/view/sort-frequency-distribution-feature-of-the-calculator/15334835/?ref=appemail>

The three measures of central tendency are mean, median, and mode. When would it make sense to use one of these measures and not the others?

### Watch This



#### MEDIA

Click image to the left or use the URL below.

URL: <http://www.ck12.org/flx/render/embeddedobject/62533>

<http://www.youtube.com/watch?v=h8EYEJ32oQ8> Khan Academy: Statistics Intro: Mean, Median, and Mode

### Guidance

With **descriptive statistics**, your goal is to describe the data that you find in a sample or is given in a problem. Because it would not make sense to present your findings as long lists of numbers, you summarize important aspects of the data. One important aspect of the data is the **measure of central tendency**, which is a measure of the “middle” value of a set of data. There are three ways to measure central tendency:

1. Use the **mean**, which is the arithmetic average of the data.
2. Use the **median**, which is the number exactly in the middle of the data. When the data has an odd number of counts, the median is the middle number after the data has been ordered. When the data has an even number of counts, the median is the arithmetic average of the two most central numbers.
3. Use the **mode**, which is the most often occurring number in the data. If there are two or more numbers that occur equally frequently, then the data is said to be bimodal or multimodal.

Calculating the mean, median and mode is straightforward. What is challenging is determining when to use each measure and knowing how to interpret the data using the relationships between the three measures.

### Example A



Five people were called on a phone survey to respond to some political opinion questions. Two people were from the zip code 94061, one person was from the zip code 94305 and two people were from 94062.

Which measure of central tendency makes the most sense to use if you want to state where the average person was from?

**Solution:** None of the measures of central tendency make sense to apply to this situation. Zip codes are categorical data rather than quantitative data even though they happen to be numbers. Other examples of categorical data are hair color or gender. You could argue that mode is applicable in a broad sense, but in general remember that mean, median, and mode can only be applied to quantitative data.

### Example B

Compute the mean, median and mode for the following numbers.

3, 5, 1, 6, 8, 4, 5, 2, 7, 8, 4, 2, 1, 3, 4, 6, 7, 9, 4, 3, 2

**Solution:**

**Mean:** The sum of all these numbers is 94 and there are 21 numbers total so the mean is  $\frac{94}{21} \approx 4.4762$ .

**Median:** When you order the numbers from least to greatest you get:

1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9

The 11<sup>th</sup> number has ten numbers to the right and ten numbers to the left so it is the median. The median is the number 4.

**Mode:** the most frequently occurring number is the number 4.

*Note: it is common practice to round to 4 decimals in AP Statistics.*

### Example C

You write a computer code to produce a random number between 0 and 10 with equal probability. Unfortunately, you suspect your code doesn't work perfectly because in your first few attempts at running the code, it produces the following numbers:

1, 9, 1, 1, 9, 2, 9, 1, 9, 9, 9, 2, 2

How would you argue using mean, median, or mode that this code is probably not producing a random number between 0 and 10 with equal probability?

**Solution:** This question is very similar to questions you will see when you study statistical inference.

First you would note that the mean of the data is 4.9231. If the data was truly random then the mean would probably be right around the number 5 which it is. This is not strong evidence to suggest that the random number generating code is broken.

Next you would note that the median of the data is 2. This should make you suspect that something is wrong. You would expect that the median is of random numbers between 0 and 10 to be somewhere around 5.

Lastly, you would note that the mode of the data is 9. By itself this is not strong data to suggest anything. Every sample will have to have at least one mode. What should make you suspicious, however, is the fact that only two other numbers were produced and were almost as frequent as the number 9. You would expect a greater variety of numbers to be produced.

### Concept Problem Revisited

In order to decide which measure of central tendency to use, it is a good idea to calculate and interpret all three of the numbers.

For example, if someone asked you how many people can sit in the typical car, it would make more sense to use mode than to use mean. With mode, you could find out that a five person car is the most frequent car driven and determine that the answer to the question is 5. If you calculate the mean for the number of seats in all cars, you will

end up with a decimal like 5.4, which makes less sense in this context.

On the other hand, if you were finding the central heights of NBA players, using mean might make a lot more sense than mode.

## Vocabulary

The **mean** is the arithmetic average of the data.

The **median** is the number in the middle of a data set. When the data has an odd number of counts, the median is the middle number after the data has been ordered. When the data has an even number of counts, the median is the average of the two most central numbers.

The **mode** is the most often occurring number in the data. If there are two or more numbers which occur equally frequently, then the data is said to be **bimodal** or **multimodal**.

With **descriptive statistics**, your goal is to describe the data that you find in a sample or is given in a problem.

With **inference statistics**, your goal is use the data in a sample to draw conclusions about a larger population.

## Guided Practice

1. Ross is with his friends and they want to play basketball. They decide to choose teams based on the number of cousins everyone has. One team will be the team with fewer cousins and the other team will be the team with more cousins. Should they use the mean, median or mode to compute the cutoff number that will separate the two teams?

2. Compute the mean, median, and mode for the following numbers.

1, 4, 5, 7, 6, 8, 0, 3, 2, 2, 3, 4, 6, 5, 7, 8, 9, 0, 6, 5, 3, 1, 2, 4, 5, 6, 7, 8, 8, 8, 4, 3, 2

3. The cost of fresh blueberries at different times of the year are:

\$2.50, \$2.99, \$3.20, \$3.99, \$4.99

If you bought blueberries regularly what would you typically pay?

### Answers:

1. Ross and his friends should use the median number of cousins as the cutoff number because this will allow each team to have the same number of players. If there are an odd number of people playing, then the extra person will just join either team or switch in later.

2. The mean is 4.6061. The median is 5. The mode is 8.

3. The word “typically” is used instead of average to allow you to make your own choice as to whether mean, median, or mode would make the most sense. In this case, mean does make the most sense. The average cost is \$3.53.

## Practice

You surveyed the students in your English class to find out how many siblings each student had. Here are your results:

0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 10, 12

1. Find the mean, median, and mode of this data.

2. Why does it make sense that the mean number of siblings is greater than the median number of siblings?

3. Which measure of central tendency do you think is best for describing the typical number of siblings?

4. So far in math you have taken 10 quizzes this semester. The mean of the scores is 88.5. What is the sum of the scores?
5. Find  $x$  if 5, 9, 11, 12, 13, 14, 16, and  $x$  have a mean of 12.
6. Meera drove an average of 22 miles a day last week. How many miles did she drive last week?
7. Find  $x$  if 2, 6, 9, 8, 4, 5, 8, 1, 4, and  $x$  have a median of 5.

Calculate the mean, median, and mode for each set of numbers:

8. 11, 15, 19, 12, 21, 34, 15, 28, 24, 15, 27, 19, 20, 13, 15
9. 3, 5, 7, 5, 5, 17, 8, 9, 11, 5, 3, 7
10. -3, 0, 5, 8, 12, 4, 2, 1, 6

Calculate the mean and median for each set of numbers:

11. 12, 88, 89, 90
12. 16, 17, 19, 20, 20, 98
13. For which of the previous two questions was the median **less than** the mean? What in the set of numbers caused this?
14. For which of the previous two questions was the median **greater than** the mean? What in the set of numbers caused this?
15. In each of the sets of numbers for problems 11 and 12, there is one number that could be considered an **outlier**. Which numbers do you think are the outliers and why? What would happen to the mean and median if you removed the outliers?

---

## 6.2 Statistical Deviation

Here you will apply what you know about mean and averages to weighted averages and expected value.

### Standard Deviation

The Standard Deviation is a measure of how spread out numbers are.

Its symbol is  $\sigma$  (the greek letter sigma)

A standard deviation close to 0 indicates that the data points tend to be very close to the **mean** (also called the expected value) of the set, while a high standard deviation indicates that the data points are spread out over a wider range of values.

View this video to learn more about standard deviation.

#### Standard Deviation

<http://www.educations.com/lesson/view/standard-deviation/15376333/?ref=appemail>

### Mean Absolute Deviation

1. The **mean absolute deviation** of a set of data is the average distance between each data value and the **mean**.
2. 1. Find the **mean**.
3. 2. Find the distance between each data value and the **mean**.

View this video to learn more about the Mean Absolute Deviation (M.A.D.)

**Mean absolute deviation** <http://youtu.be/PwsXncM2pas>

## 6.3 Five Number Summary, Box and Whisker Plots, and Modified Box Plots and Outliers

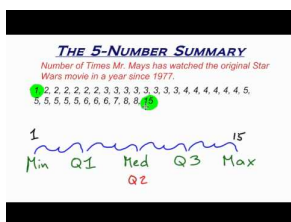
Here you will calculate quartiles and produce five number summaries for data sets.

When given a long list of numbers, it is useful to summarize the data. One way to summarize the data is to give the lowest number, the highest number and the middle number. In addition to these three numbers it is also useful to give the median of the lower half of the data and the median of the upper half of the data. These five numbers give a very concise summary of the data.

What is the five number summary of the following data?

0, 0, 1, 2, 63, 61, 27, 13

### Watch This



### MEDIA

Click image to the left or use the URL below.

URL: <http://www.ck12.org/flx/render/embeddedobject/62539>

<http://www.youtube.com/watch?v=XDS5TgZ4CJA> The 5 Number Summary

In data sets that are large enough, you can divide the numbers into four parts called quartiles. The quartiles of interest are the first quartile, , the second quartile, , and the third quartile . The second quartile, , is defined to be the median of the data. The first quartile, , is defined to be the median of the lower half of the data. The third quartile, , is similarly defined to be the median of the upper half of the data.

These three numbers in addition to the minimum and maximum values are the five number summary. Note that there are variations of the five number summary that you can study in a statistics course.

### Example A

Compute the five number summary for the following data.

2, 7, 17, 19, 25, 26, 26, 32

**Solution:** There are 8 observations total.

- Lowest value (minimum) : 2
- $Q1 : \frac{7+17}{2} = 12$  (Note that this is the median of the first half of the data - 2, 7, 17, 19)
- $Q2 : \frac{19+25}{2} = 22$  (Note that this is the median of the full set of data)
- $Q3 : 26$  (Note that this is the median of the second half of the data - 25, 26, 26, 32)
- Upper value (maximum) : 32

### Example B

Compute the five number summary for the following data:

4, 8, 11, 11, 12, 14, 16, 20, 21, 25

**Solution:** There are 10 observations total.

- Lowest value (minimum) : 4
- $Q1$  : 11 (*Note that this is the median of the first half of the data - 4, 8, 11, 11, 12*)
- $Q2$  :  $\frac{12+14}{2} = 13$  (*Note that this is the median of the full set of data*)
- $Q3$  : 20 (*Note that this is the median of the second half of the data - 14, 16, 20, 21, 25*)
- Upper value (maximum) : 25

### Example C

Compute the five number summary for the following data:

3, 7, 10, 14, 19, 19, 23, 27, 29

**Solution:** There are 9 observations total. To calculate  $Q1$  and  $Q3$ , you should include the median in both the lower half and upper half calculations.

- Lowest value (minimum) : 3
- $Q1$  : 10 (*this is the median of 3, 7, 10, 14, 19*)
- $Q2$  : 19
- $Q3$  : 23 (*this is the median of 19, 19, 23, 27, 29*)
- Upper value (maximum) : 29

### Concept Problem Revisited

To compute the five number summary, it helps to order the data.

0, 0, 1, 2, 13, 27, 61, 63

- Since there are 8 observations, the median is the average of the 4<sup>th</sup> and 5<sup>th</sup> observations:  $\frac{2+13}{2} = 7.5$
- The lowest observation is 0.
- The highest observation is 63.
- The middle of the lower half is  $\frac{0+1}{2} = 0.5$
- The middle of the upper half is  $\frac{27+61}{2} = 44$

The five number summary is 0, 0.5, 7.5, 44, 63

### Vocabulary

The **rank** of an observation is the number of observations that are less than or equal to the value of that observation.

Data is divided into four parts by the **first quartile** ( $Q1$ ), **second quartile** ( $Q2$ ) and **third quartile** ( $Q3$ ). The **second quartile** is also known as the median.

### Guided Practice

1. Create a set of data that meets the following five number summary:

{2, 5, 9, 18, 20}

2. Compute the five number summary for the following data:

1, 1, 1, 2, 2, 3, 3, 3, 4, 4, 5, 5, 5, 6, 6, 7, 8, 9, 10, 15

3. Compute the five number summary for the following data:

1, 4, 96, 356, 2557, 9881, 14420, 20100

**Answers:**

1. Suppose there are 8 data points. The lowest point must be 2 and the highest point must be 20. The middle two points must average to be 9 so they could be 8 and 10. The second and third points must average to be 5 so they could be 4 and 6. The sixth and seventh points need to average to be 18 so they could be 18 and 18. Here is one possible answer:

2, 4, 6, 8, 10, 18, 18, 20

2. There are 20 observations.

- Lower : 1
- $Q1 : \frac{2+3}{2} = 2.5$
- $Q2 : \frac{4+5}{2} = 4.5$
- $Q3 : \frac{6+7}{2} = 6.5$
- Upper : 15

3. There are 8 observations.

- Lower : 1
- $Q1 : \frac{4+96}{2} = 50$
- $Q2 : \frac{356+2557}{2} = 1456.5$
- $Q3 : \frac{9881+14420}{2} = 12150.5$
- Upper: 20100

**Box and Whisker Plot**

We can show all the important values in a "Box and Whisker Plot", like this:

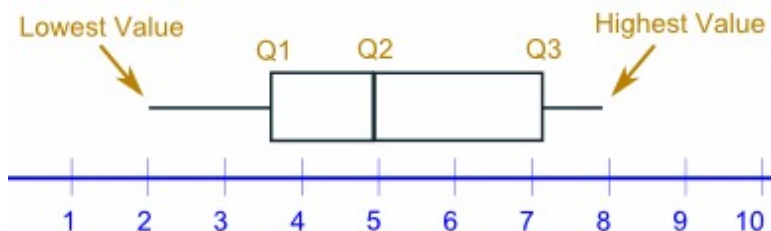


FIGURE 6.1

To Construct a Box and Whisker Plot on the Graphing Calculator, view this video:

**Box & Whisker Plots** <http://www.educations.com/lesson/view/box-whisker-plots-on-the-calculator/15377505/?ref=appemail>

**Outliers and Interquartile Range**

The "interquartile range", abbreviated "IQR", is just the width of the box in the box-and-whisker plot. That is,  $IQR = Q_3 - Q_1$ . The IQR can be used as a measure of how spread-out the values are. An outlier is any value that lies more

than one and a half times the length of the box from either end of the box. View this video to learn more regarding outliers and interquartile range:

**Outliers and the Interquartile Range** <http://www.educreations.com/lesson/view/outliers-interquartile-range/15377797/?ref=appemail>

The modified box-and-whisker plot will not plot outliers as part of the box-and-whisker. The outliers are plotted as individual points beyond the whisker in an attempt to give a more accurate picture of the dispersion of the data.

The standard box-and whisker plot includes ALL data points, including what are called outliers. Outliers are points that are far left or far right of the data set and may detract from a representative picture of the data. View this video to learn to create a modified box plot on your graphing calculator:

**Modified Box Plots** [http://youtu.be/rkJX0eA-E\\_M](http://youtu.be/rkJX0eA-E_M)

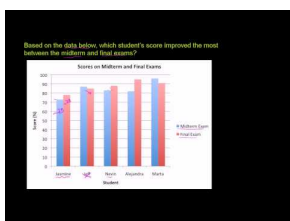


## 6.4 Graphic Displays of Data - Bar Graphs, Pie Charts, Histograms, Cumulative Histograms, Stem and Leaf

Here you will display data using bar charts, histograms, pie charts and boxplots.

Two common types of graphic displays are bar charts and histograms. Both bar charts and histograms use vertical or horizontal bars to represent the number of data points in each category or interval. The main difference graphically is that in a bar chart there are spaces between the bars and in a histogram there are not spaces between the bars. Why does this subtle difference exist and what does it imply about graphic displays in general?

### Watch This

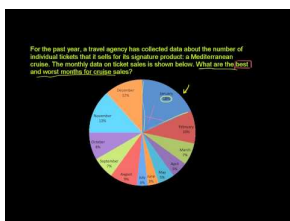


#### MEDIA

Click image to the left or use the URL below.

URL: <http://www.ck12.org/flx/render/embeddedobject/62546>

<http://www.youtube.com/watch?v=kiQ6MUQZHSs> Khan Academy: Reading Bar Graphs



#### MEDIA

Click image to the left or use the URL below.

URL: <http://www.ck12.org/flx/render/embeddedobject/62548>

<http://www.youtube.com/watch?v=4JqH55rLGKY> Khan Academy: Reading Pie Graphs

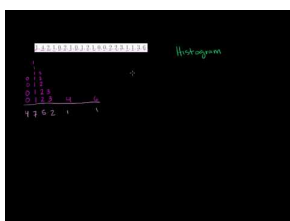


#### MEDIA

Click image to the left or use the URL below.

URL: <http://www.ck12.org/flx/render/embeddedobject/62550>

<http://www.youtube.com/watch?v=b2C9I8HuCe4> Khan Academy: Reading Box-and-Whisker Plots



#### MEDIA

Click image to the left or use the URL below.

URL: <http://www.ck12.org/flx/render/embeddedobject/62552>

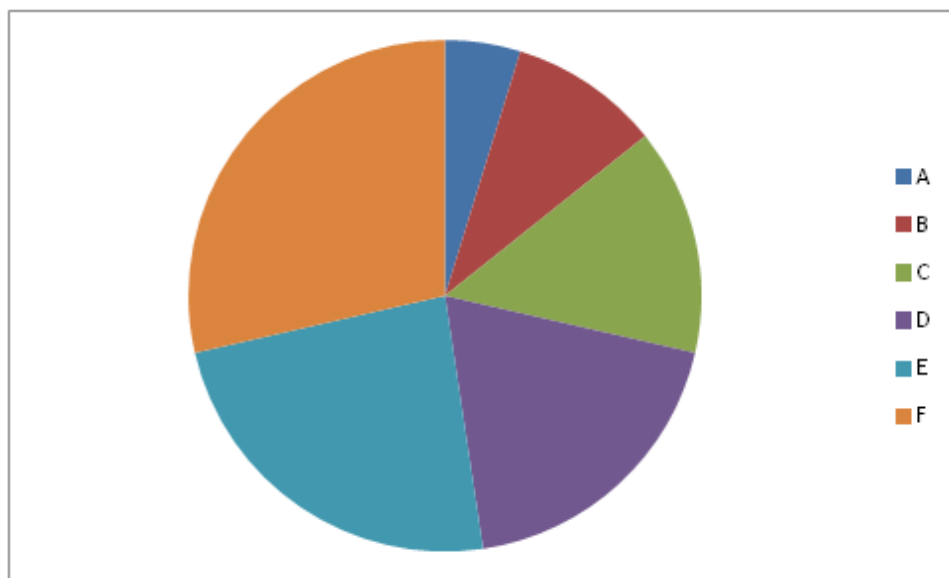
<http://www.youtube.com/watch?v=4eLJGG2Ad30> Khan Academy: Histogram

## Guidance

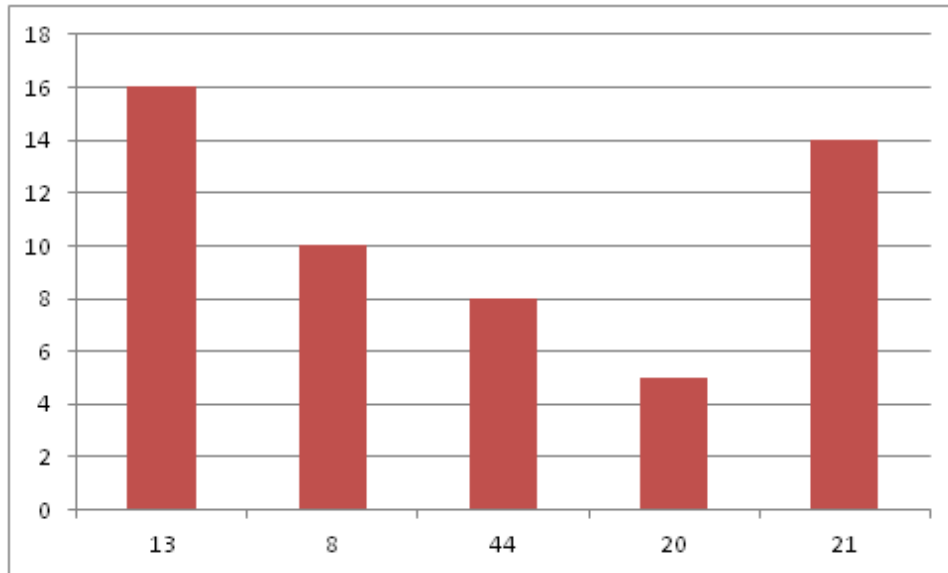
It is often easier for people to interpret relative sizes of data when that data is displayed graphically. There are a few common ways of displaying data graphically that you should be familiar with.

1) A **pie chart** shows the relative proportions of data in different categories. Pie charts are excellent ways of displaying categorical data with easily separable groups. The following pie chart shows six categories labeled  $A - F$ . The size of each pie slice is determined by the central angle. Since there are  $360^\circ$  in a circle, the size of the central angle  $\theta_A$  of category  $A$  can be found by:

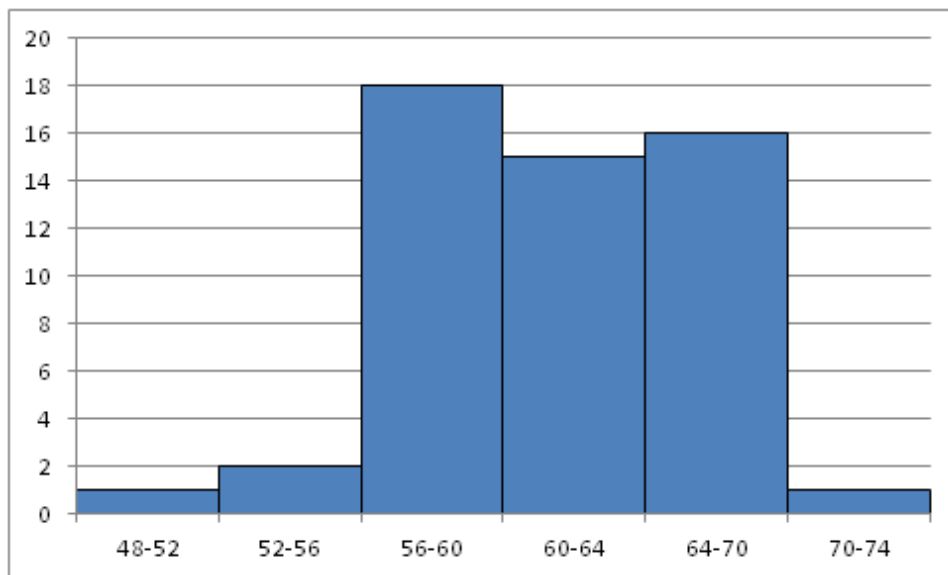
$$\frac{\theta_A}{360} = \frac{\# \text{ data points in category } A}{\text{Total number of data points}}$$



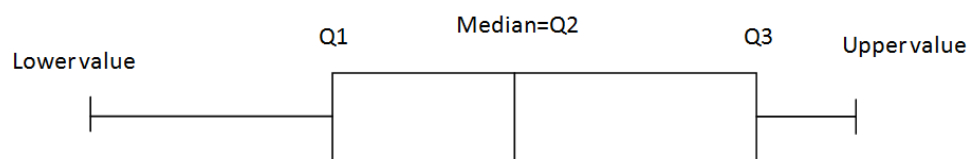
2) A **bar chart** displays frequencies of categories of data. The bar chart below has 5 categories, and shows the TV channel preferences for 53 adults. The horizontal axis could have also been labeled *News*, *Sports*, *Local News*, *Comedy*, *Action Movies*. The reason why the bars are separated by spaces is to emphasize the fact that they are categories and not continuous numbers. For example, just because you split your time between channel 8 and channel 44 does not mean on average you watch channel 26. Categories can be numbers so you need to be very careful.



3) A **histogram** displays frequencies of quantitative data that has been sorted into intervals. The following is a histogram that shows the heights of a class of 53 students. Notice the largest category is 56-60 inches with 18 people.



4) A **boxplot** (also known as a **box and whiskers plot**) is another way to display quantitative data. It displays the five number summary (minimum,  $Q1$ , median,  $Q3$ , maximum). The box can either be vertically or horizontally displayed depending on the labeling of the axis. The box does not need to be perfectly symmetrical because it represents data that might not be perfectly symmetrical.

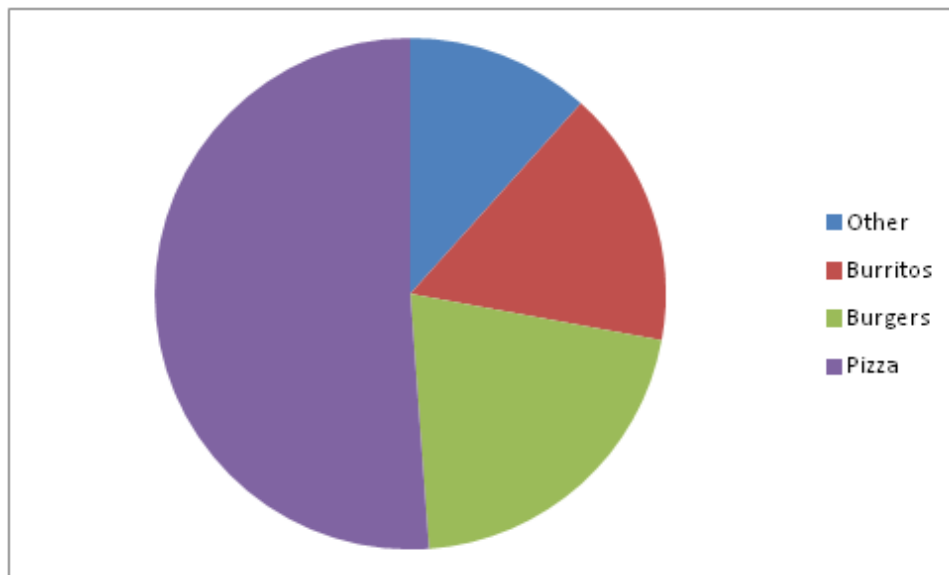


### Example A

Create a pie chart to represent the preferences of 43 hungry students.

- Other – 5
- Burritos – 7
- Burgers – 9
- Pizza – 22

**Solution:**

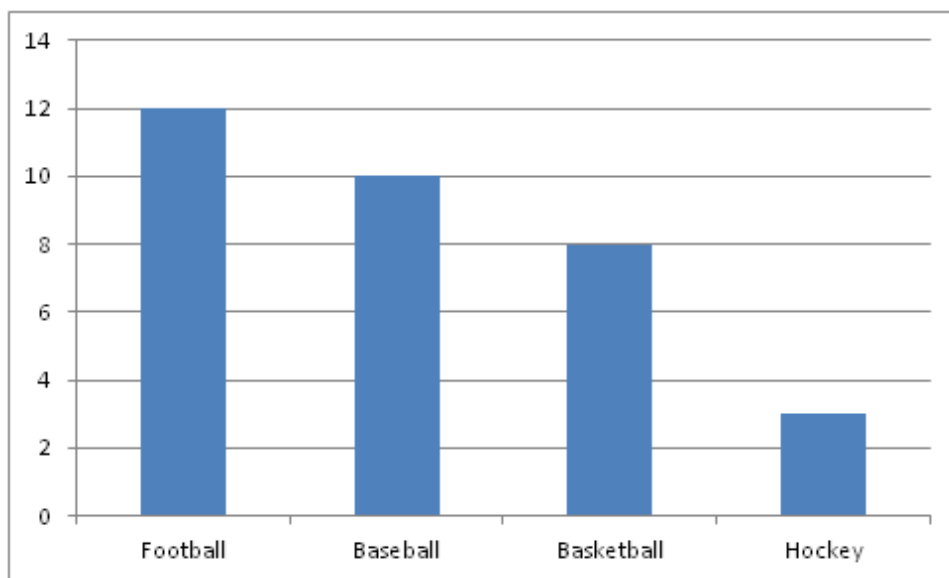


### Example B

Create a bar chart representing the preference for sports of a group of 23 people.

- Football – 12
- Baseball – 10
- Basketball – 8
- Hockey – 3

**Solution:**

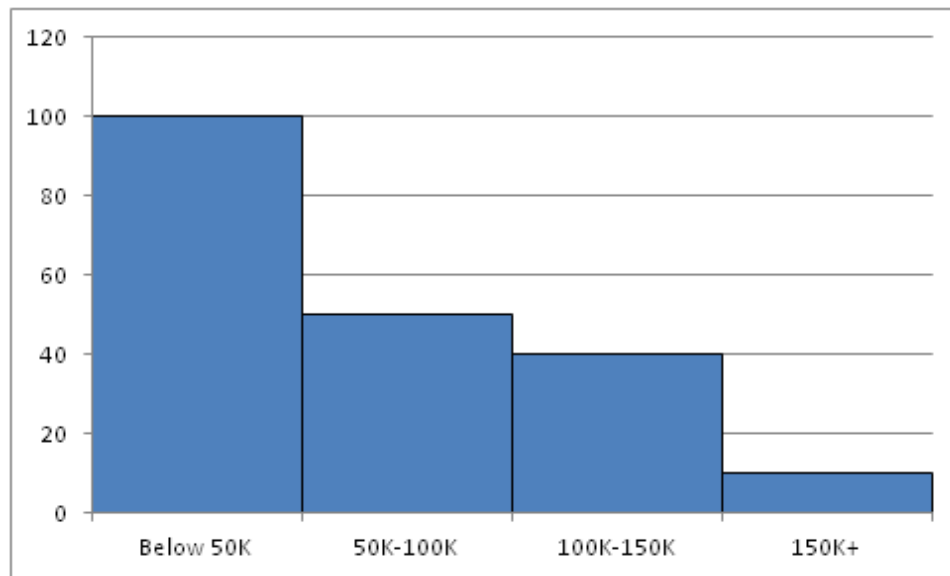


**Example C**

Create a histogram for the income distribution of 200 million people.

- Below \$50,000 is 100 million people
- Between \$50,000 and \$100,000 is 50 million people
- Between \$100,000 and \$150,000 is 40 million people
- Above \$150,000 is 10 million people

**Solution:**



**View this video to learn to create a frequency distribution on a graphing calculator.**

**Frequency Distributions and Histograms** <http://www.educreations.com/lesson/view/frequency-distributions-histograms/15378859/?ref=appemail>

**Cumulative Frequency Histogram:**

A **cumulative frequency** is the running total of the **frequencies**. View this video to learn more regarding cumulative frequency histograms. **Cumulative Frequency Histograms** <http://www.educreations.com/lesson/view/cumulative-frequency-histograms/16865789/?ref=app>

**Stem and Leaf Diagrams**

A **Stem and Leaf Plot** is a special table where each data value is split into a "stem" (the first digit or digits) and a "leaf" (usually the last digit).

**For example:**

**2.3, 2.5, 2.5, 2.7, 2.8 3.2, 3.6, 3.6, 4.5, 5.0**

And here is the stem-and-leaf plot:

TABLE 6.1:

Stem	Leaf
2	3
	5
	5
	7
	8
3	2
	6
	6
4	5
5	0

**View this video to learn more about Stem and Leaf Diagrams:**

**Stem and Leaf Diagrams** <http://www.educreations.com/lesson/view/stem-leaf-diagrams/15327697/?ref=appemail>

### Concept Problem Revisited

The reason for the space in bar charts but no space in histograms is bar charts graph categorical variables while histograms graph quantitative variables. It would be extremely improper to forget the space with bar charts because you would run the risk of implying a spectrum from one side of the chart to the other. Note that in the bar chart where TV stations were shown, the station numbers were not listed horizontally in order by size. This was to emphasize the fact that the stations were categories.

### Vocabulary

A **categorical variable** is a variable that can take on one of a limited number of values. Examples of categorical variables are tv stations, the state someone lives in, and eye color.

A **quantitative variable** is a variable that takes on numerical values that represent a measureable quantity. Examples of quantitative variables are the height of students or the population of a city.

A **bar chart** is a graphic display of categorical variables that uses bar to represent the frequency of the count in each category.

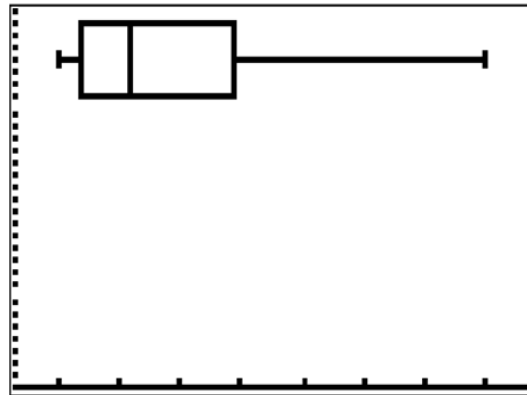
A **histogram** is a graphic display of quantitative variables that uses bars to represent the frequency of the count of the data in each interval.

A **pie chart** is a graphic display of categorical data where the relative size of each pie slice corresponds to the frequency of each category.

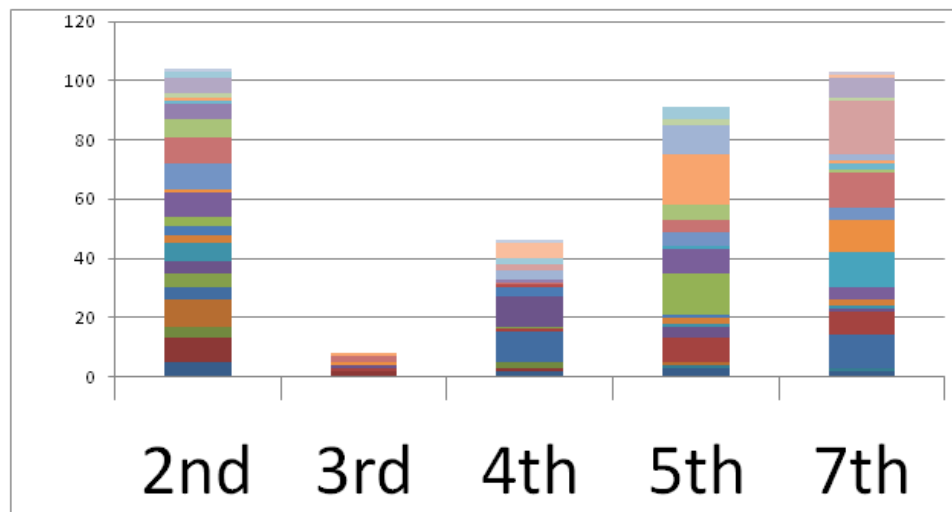
A **boxplot** is a graphic display of quantitative data that demonstrates the five number summary.

### Guided Practice

1. Create a boxplot of the following numbers in your calculator.  
8.5, 10.9, 9.1, 7.5, 7.2, 6, 2.3, 5.5
2. Identify the interesting characteristics of the following boxplot.



3. Interpret the following bar chart that represents the number of tardy students in 5 class periods over the course of a year.

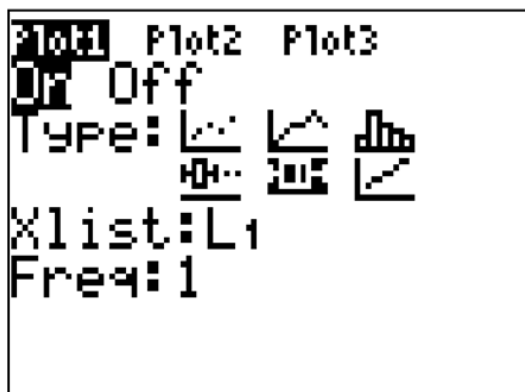


### Answers:

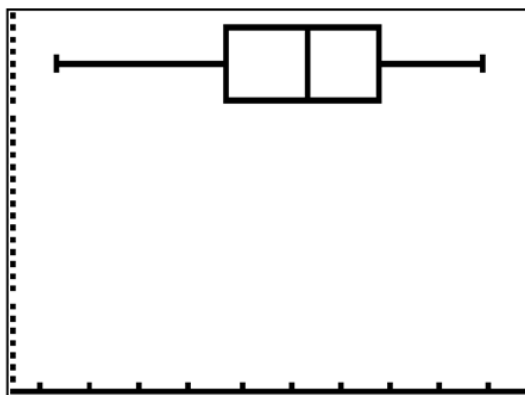
1. Enter the data into  $L_1$  by going into the Stat menu.

L1	L2	L3	2
8.5	██████	-----	
10.9			
9.1			
7.5			
7.2			
6			
2.3			
L2(1)=			

Then turn the statplot on and choose boxplot.



Use Zoomstat to automatically center the window on the boxplot.



2. The lower bound,  $Q1$  and  $Q2$  all seem to be relatively close together.  $Q3$  seems to be stretched a little to the right and the upper bound is significantly stretched to the right.
3. The bar chart has 5 categories representing each of the five periods. Within each category there are bands of different colors. Each band represents the number of times an individual student was tardy. For periods 5 and periods 7 there seem to be fewer students who were tardy more often. In period 2 there seems to be more students tardy a handful of times each.

### Practice

1. What types of graphs show categorical data?
2. What types of graphs show quantitative data?

A math class of 30 students had the following grades:

**TABLE 6.2:**

Grade	Number of Students with Grade
A	10
B	10
C	5
D	3
F	2



3. Create a bar chart for this data.

4. Create a pie chart for this data.

5. Which graph do you think makes a better visual representation of the data?

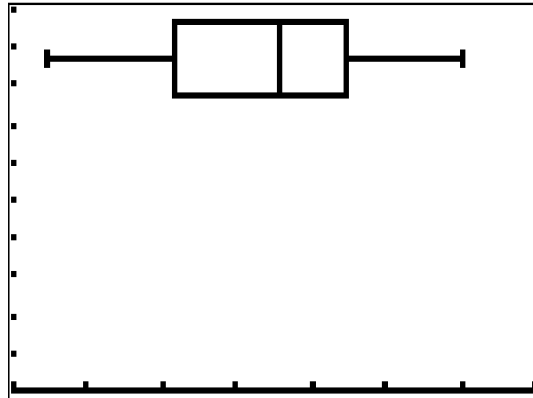
A set of 20 exam scores is 67, 94, 88, 76, 85, 93, 55, 87, 80, 81, 80, 61, 90, 84, 75, 93, 75, 68, 100, 98

6. Create a histogram for this data. Use your best judgment to decide what the intervals should be.

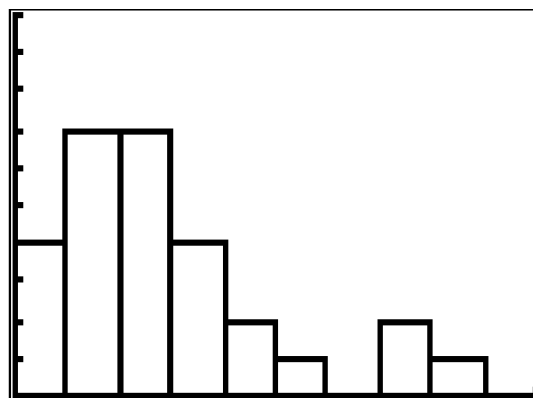
7. Find the five number summary for this data.

8. Use the five number summary to create a boxplot for this data.

9. Describe the data shown in the boxplot below.



10. Describe the data shown in the histogram below.



A math class of 30 students has the following eye colors:

**TABLE 6.3:**

Grade	Number of Students with Grade
Brown	20
Blue	5
Green	3
Other	2

11. Create a bar chart for this data.

12. Create a pie chart for this data.

13. Which graph do you think makes a better visual representation of the data?
14. Suppose you have data that shows the breakdown of registered republicans by state. What types of graphs could you use to display this data?
15. From which types of graphs could you obtain information about the spread of the data? Note that spread is a measure of how spread out all of the data is.

---

## 6.5 Other Types of Averages

Here you will calculate population variance, sample variance and standard deviation from univariate data.

### Trimmed Means

A method of averaging that removes a small percentage of the largest and smallest values before calculating the mean. After removing the specified observations, the trimmed mean is found using an arithmetic averaging formula.

The trimmed mean looks to reduce the effects of outliers on the calculated average.

#### To Compute a 5% Trimmed Mean:

- 1) Order the data from smallest to largest.
- 2) Delete the bottom 5% of the data.
- 3) Delete the top 5% of the data.
- 4) Compute the mean of the remaining 90% of the data.

View this video for examples of calculating the 5 % Trimmed Mean and the 10 % Trimmed Mean for data: **Trimmed Means** <http://www.educreations.com/lesson/view/trimmed-means/15376032/?ref=appemail>

#### To Compute a 10% Trimmed Mean:

- 1) Order the data from smallest to largest.
- 2) Delete the bottom 10% of the data.
- 3) Delete the top 10% of the data.
- 4) Compute the mean of the remaining 80% of the data.

### Weighted Average

- Not everything counts the same. For example, you final exam score "weighs more" than a regular class test does.

#### Definition: Weighted Average

An average in which each quantity to be averaged is assigned a weight. These weightings determine the relative importance of each quantity on the average. Weightings are the equivalent of having that many like items with the same value involved in the average.

### 'Weighted Average'

To demonstrate, let's take the value of letter tiles in the popular game Scrabble.

Value:	10	8	5	4	3	2	1	0
Occurrences:	2	2	1	10	8	7	68	2

To average these values, do a weighted average using the number of occurrences of each value as the weight. To calculate a weighted average:

1. Multiply each value by its weight. (*Ans: 20, 16, 5, 40, 24, 14, 68, and 0*)
2. Add up the products of value times weight to get the total value. (*Ans: Sum=187*)
3. Add the weight themselves to get the total weight. (*Ans: Sum=100*)
4. Divide the total value by the total weight. (*Ans:  $187/100 = 1.87 = \text{average value of a Scrabble tile}$* )

To learn more regarding weighted average, view the following video **Weighted Average** <http://www.educreations.com/lesson/view/weighted-average/15375811/?ref=appemail>

## 6.6 Percentile Rank

Here you will define the standard normal distribution and learn how standard deviation and the area under the curve are connected.

View this video to receive an introduction to the concept of Percentile Rank: **Percentile rank** <http://www.educations.com/lesson/view/percentile-rank/16342497/?ref=app>

Percentiles: The quartiles and the median are special cases of **percentiles** for a data set. In general, the  $k$ th *percentile* is a number that has  $k\%$  of the data values at or below it and  $(100 - k)\%$  of the data values at or above it. The lower quartile, median, and upper quartile are also the 25<sup>th</sup> percentile, 50<sup>th</sup> percentile, and 75<sup>th</sup> percentile, respectively. If you are told that you scored at the 90<sup>th</sup> percentile on a standardized test (like the SAT), it indicates that 90% of the scores were at or below your score, while 10% were at or above your score.

Use the following data values:

284	586	987	412	256	541	312	251	444	695
421	789	234	190	387	561	631	821	520	813

Which number closest represents the 42<sup>nd</sup> percentile?

### To find the value for a percentile ranking

- 1) Sort the data in ascending order
- 2) Calculate: 
$$\frac{\text{percentile ranking}}{100} \times \text{the number of data values}$$
- 3) Round up to the nearest integer
- 4) That value calculated in step 3 is the “place” in which the answer lies.

Ex)  $42/100 \times 20 = 8.4 \rightarrow 9$

The data value in the 9<sup>th</sup> spot is the answer.      ANSWER = 421

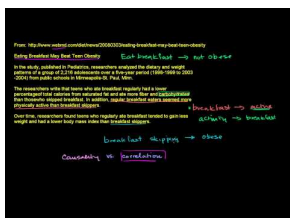
## 6.7 Linear Correlation

Here you will begin to work with bivariate data as you learn about linear correlation, correlation coefficients, and regression.

Statistics is largely concerned with how a change in one variable relates to changes in a second variable. Bivariate data is two lists of data that are paired up. Is there any relationship between the following data? If there is, does it mean that doctors cause cancer?

Number of Doctors	27	30	36	60	81	90	156	221	347
Cancer Rate	0.02	0.07	0.16	0.20	0.43	0.87	1.21	2.80	3.91

### Watch This



### MEDIA

Click image to the left or use the URL below.

URL: <http://www.ck12.org/flx/render/embeddedobject/62997>

<http://www.youtube.com/watch?v=ROpbdO-gRUo> Khan Academy: Correlation vs. Causality

### Guidance

A **scatterplot** creates an  $(x,y)$  point from each data pair. When making a scatterplot, you can try to assign the independent variable to  $x$  and the dependent variable to  $y$ ; however, it will often not be obvious which variable is the dependent variable, so you will just have to pick one.

View this video to learn how to create a scatterplot on the graphing calculator: **How to create a scatterplot on the graphing calculator** [http://youtu.be/Zjj13\\_pKtwM](http://youtu.be/Zjj13_pKtwM)

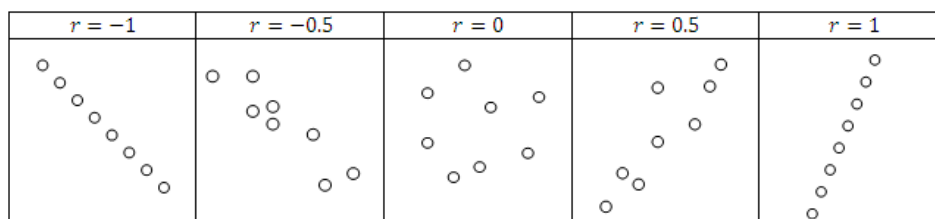
View this video to learn how to interpret a scatterplot **Interpreting scatterplots** [http://youtu.be/PE\\_BpXTyKCE](http://youtu.be/PE_BpXTyKCE)

Once you plot the data and zoom appropriately you will see the points scattered about. Sometimes there will be a clear linear relationship and sometimes it will appear random. The **correlation coefficient**,  $r$ , is a number that quantifies two aspects of the relationship between the data:

- The correlation coefficient is either negative, zero or positive. This tells you whether the data is negatively correlated, uncorrelated or positively correlated.

- The correlation coefficient is a number between  $-1 \leq r \leq 1$  indicating the strength of correlation. If  $r = 1$  or  $r = -1$  then the data is perfectly linear. Note that a perfectly linear relationship includes lines with slopes other than 1.

Consider the examples below to see what different correlation coefficients will look like in data:



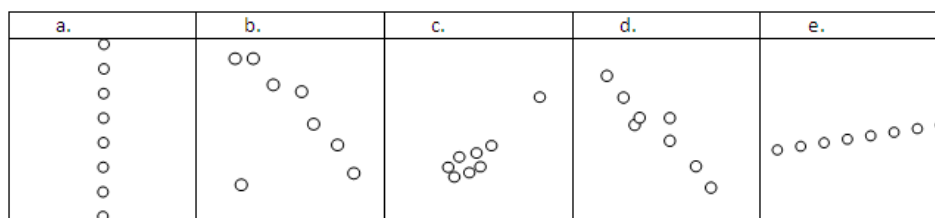
Here are three additional videos to help you to understand correlation: **Correlation** <http://www.educreations.com/lesson/view/correlation/651033/?ref=app>

**Types of Correlation** <http://youtu.be/CWnfwZRAuaY>

**Examples of Correlation** <http://youtu.be/AdEE07PWeEA>

### Example A

Estimate the correlation coefficient for the following scatterplots.



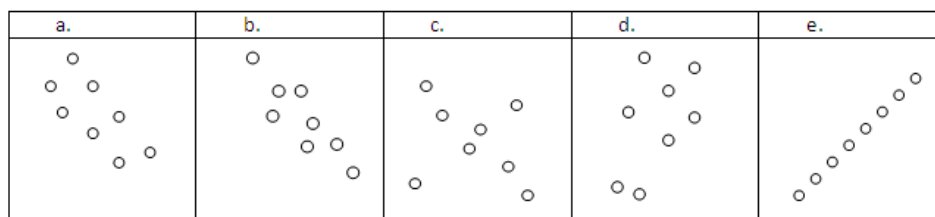
### Solution:

- $r \approx 0$ . Because the height ( $y$ ) does not seem to be dependent on the  $x$ , the data is uncorrelated. Another way to see this is that the slope appears to be undefined.
- $r \approx -0.7$ . If the solo point in the bottom left is an outlier, you could choose to not include it in the data. Then, the  $r$  value would be closer to -1.

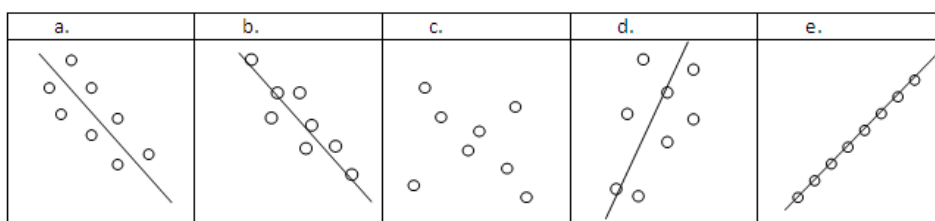
- c.  $r \approx +0.8$ . The clump of data seems to be slightly positive correlated and the single point in the upper left has a strong effect indicating positive slope.
- d.  $r \approx -0.8$ . The data seems to be fairly strongly negatively correlated.
- e.  $r \approx 1$ . The data seems to be perfectly linearly correlated.

### Example B

Estimate the regression line through the following scatterplots.



**Solution:** Visualize and sketch the “line of best fit” for each set of points.



Note that in part a, the regression line does not touch any point. Instead, it captures the general trend of the data. In part c, the correlation is not high enough in any direction to produce a regression line. The calculator may give a regression line for scatterplots that look like part c, but you need to be very skeptical that there is actually a relationship between the two variables.

In PreCalculus you will not learn how to calculate the correlation coefficient (you will if you take future statistics courses!). For now, the calculator will calculate it for you and your job will be to interpret the result. See Example C.

If the data is sufficiently linear, then your calculator can perform a regression to produce the equation of a line that attempts to model the trend of the data. The regression line may actually pass through all, some or none of the data points. This regression line is represented in statistics by:

$$y = ax + b$$

a represents the slope of the line

b represents the y-intercept of the line

**To do Linear Regression on the graphing calculator:**

STAT

CALC



**4: LinReg (ax b)****Xlist:** L1 (2<sup>nd</sup> 1 – L1)**Ylist:** L2 (2<sup>nd</sup> 2 – L2)**FreqList:** Keep blank**Store RegEQ:** Y1 (ALPHA F4 Y1 ENTER)**Calculate:** Press Enter**Example C**

Use your calculator to perform a linear regression on the following data. Then, predict the height of someone who has shoe size 9.

**TABLE 6.5:**

Shoe Size	Height (in)
11	70
8.5	70
10	72
8	65
7	64

**Solution:** First enter the data.

L1	L2	L3	2
11	70	-----	
8.5	70		
10	72		
8	65		
7	64		
-----	-----		
L2(6) =			

Next perform the regression. Notice that the calculator can perform linear regression in two ways that are essentially the same.

**USE OPTION 4: LinReg(ax b)**

```

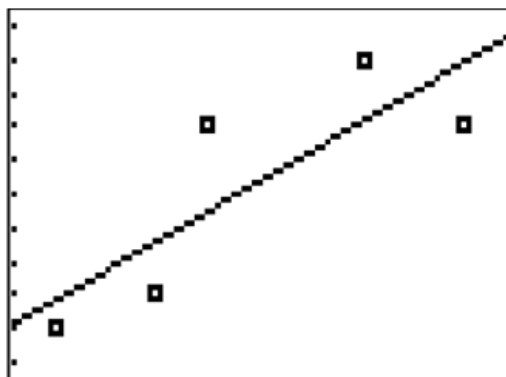
EDIT [2nd] [MODE] TESTS
2:2-Var Stats
3:Med-Med
4:LinReg(ax+b)
5:QuadReg
6:CubicReg
7:QuartReg
8:LinReg(a+bx)

```

Notice that the  $r$  value is about 0.8. This indicates that there is a fairly strong positive correlation between shoe size and height. *If your calculator does not display the  $r$  and  $r^2$  lines then you need to go into the catalog and run the program "DiagnosticOn". This will enable the display of the correlation coefficient.*

If you are not shown  $r$  and  $r^2$  on your calculator, view this video: **Getting R and R<sup>2</sup> on your calculator if you don't have it already** <http://youtu.be/HK-gyKwNR6M>

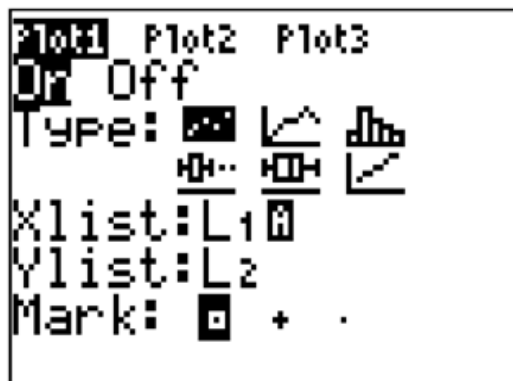
You can then graph the scatterplot and the regression line by pressing GRAPH

**Concept Problem Revisited**

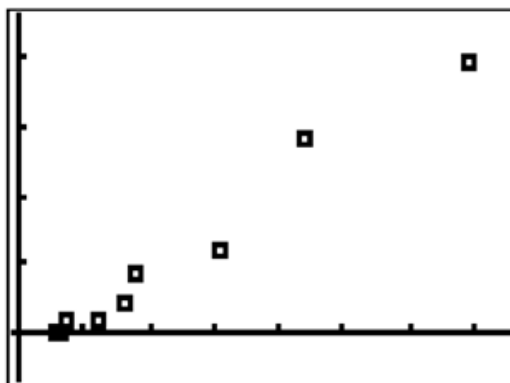
Enter the data onto lists in your calculator:

L1	L2	L3	2
27	.02	-----	
30	.07		
36	.16		
60	.2		
81	.43		
90	.87		
156	1.21		
L2(1)=.02			

Turn the [STAT PLOT] on that compares the two lists of data:



You should note that the data is extremely linear with a positive correlation coefficient:



A naïve conclusion would be to say that doctors cause cancer. One of the most misunderstood concepts in statistics is that correlation does not imply causation. Just because there is a correlation between the number of doctors and the cancer rate doesn't mean that the number of doctors *causes* the cancer. There are dozens of reasons why more doctors might correlate with higher cancer rates. In general, remember that correlation is not the same as causation. Be careful before making any conclusions about change in one variable *causing* change in another variable.

## Vocabulary

A **scatterplot** creates an  $(x,y)$  point from each data pair.

**Bivariate data** is two sets of data that are paired.

The **correlation coefficient**,  $r$ , is a number in the interval  $[-1, 1]$ . It indicates the strength of the correlation between two variables.

## Guided Practice

- The data below represents the average number of working words in an elementary student's vocabulary as it relates to their shoe size. Perform a linear regression that models the data.

---

Shoe Size	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
Vocabulary	135	198	325	501	411	354	317	891

2. Use the equation from Guided Practice 1 to predict the vocabulary for someone who has a 1.0 shoe size. Does this prediction seem reasonable given the data? Why or why not?
3. Shaquille O'Neal has size 23 shoes. What, if anything can you infer about his vocabulary? Does a larger shoe size cause a larger vocabulary?

## 6.8 Other Regression Types Regression

Here you will use regression on a variety of different types of data to make reasonable predictions.

For the Hyundai Elantra, a study was done to compare the speed  $x$  (in mph) with the gas mileage  $y$  (in miles per gallon).

speed	mpg	speed	mpg	speed	mpg
15	22.3	35	28.8	55	30.4
20	25.5	40	30.0	60	28.8
25	27.5	45	29.9	65	27.4
30	29.0	50	30.2	70	25.3

- Use a graphing utility to create a scatterplot.
- Use the regression feature to find a linear model that fits the data
- How well does the equation best fit the data?

Enter the speeds under L1 and the mpg under L2.

STAT EDIT

2<sup>ND</sup> STAT PLOT

ON

TYPE 1

Xlist: L1

Ylist: L2

Zoom

9: ZoomStat

- Use the regression feature to find a linear model that fits the data

STAT CALC

4:LinReg(ax+b)

Xlist: L1

Ylist: L2

FreqList: Keep Blank

Store ReqEQ: ALPHA F4 Y1

Calculate

$$y = ax + b$$

$$a = .0491608392$$

$$b = 25.83566434$$

$$r = .358936349$$

$$y = .05x + 25.8 \quad (\text{but do not round off})$$

(c) How well does the equation best fit the data ?

Since  $r$  is .36, NOT very well.

If you look at the scatterplot, what type of pattern do the points seem to follow?

A QUADRATIC PATTERN

### TO PERFORM A QUADRATIC REGRESSION OF DATA

-

STAT

CALC

5:QuadReg

Xlist: L1

Ylist: L2

FreqList: Keep Blank

Store RegEQ: ALPHA F4 Y2 (normally Y1 but  
the linear eq. we did is Y1)

Calculate ENTER

$$y = ax^2 + bx + c$$

$$a = -.0081968032$$

$$b = .7458891109$$

$$c = 13.47215285$$

$$R^2 = .964557828 \quad (\text{coefficient of determination})$$

### Two notes

1) The equation  $y = -.008x^2 + .746x + 13.472$  is the quadratic equation of best fit (Don't round off – the equation is stored next to Y2 – normally Y1)

2) You must radicalize  $R^2$  to get the r value.

$$R^2 = .964557828 \quad \rightarrow \quad r = .9821190498$$

How well does the equation best fit the data now? **VERY VERY WELL**

You are going to be asked to perform regressions and are going to be asked which is the “better fit”.

The regression with the greater r-value (regression coefficient) is the better fit.

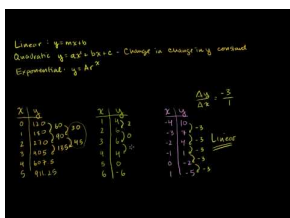
### Note:

There are many different types of mathematical regressions.

Linear, Quadratic, Cubic ( $x^3$ ), Quartic ( $x^4$ ), logarithm, exponential, power, trigonometric, ETC.

Linear correlation is the simplest type of relationship between two variables. Your calculator has the power to use a variety of different function families to find other relationships and create many different types of models. How do you choose which function family is best for a given situation?

### Watch This



### MEDIA

Click image to the left or use the URL below.

URL: <http://www.ck12.org/flx/render/embeddedobject/60152>

<http://www.youtube.com/watch?v=CxEFOozrMSE> Khan Academy: Linear, Quadratic, and Exponential Models

### Guidance

Once you understand how to do linear regression with your calculator, you already know the technical mechanics to perform other regressions in the [STAT] [CALC] menu. The most common regressions correspond to the function families.

- QuadReg - Quadratic function family
- CubicReg - Cubic function family
- QuartcReg - Quartic function family or  $4^{th}$  degree polynomial
- LnReg - Natural Log function family
- ExpReg - Exponential function family
- PwrReg - Power function family
- Logistic - Logistic function family
- SinReg - Sinusoidal function family.

When you perform these types of regressions, it will be incredibly important for you to interpret and explain parts of the graph. Here are some points to keep in mind:

1. The y-intercept may have a particular meaning that may or may not be reasonable.
2. When you use your model to make predictions it is important for you to remember the relevant domain of your model. If your data is about elementary school students then it might extend to middle and high school students, but it might not.
3. The calculator may produce a correlation coefficient for each of these non-linear regressions, but you should be very careful. Technically, the correlation coefficient is only supposed to be calculated with linear regression, so the calculator is doing some fancy linearization to produce it. You can learn more about this process in future statistics courses.

In general, at this point you should use your best judgment when choosing a function family to model a given set of data and deciding how good a fit the model is based on context.

### Example A

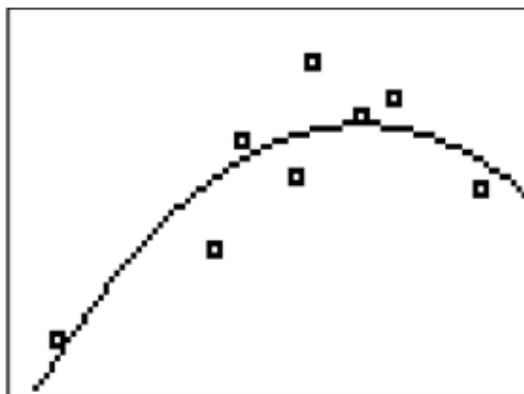
Given the following data about SAT scores and number of hours slept the night before, use an appropriate function family to produce a reasonable model. Defend your choice of function families.

TABLE 6.7:

# Hours Slept	SAT Score
8.5	1840
10.9	1510
9.1	1900
7.5	2070
7.2	1550
6.0	1720
2.3	840
5.5	1230

**Solution:** Let  $x$  be the number of hours slept and  $y$  be the SAT score.

After plotting the points, you should choose a function family to use as a model. In this case, it would be appropriate to try a quadratic relationship.



A quadratic model makes sense because there seems to be a peak in the model and in the data around 8.5 hours of sleep. It makes sense that someone who does not get enough sleep will do worse and someone who gets too much



sleep might also do worse.

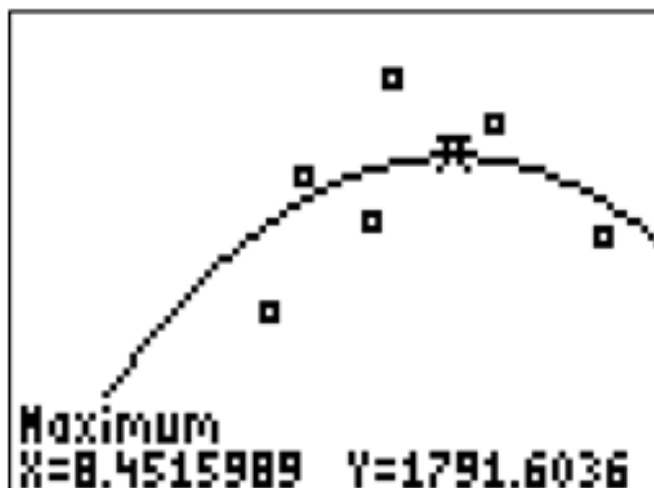
### Example B

Using the model from Example A, answer the following questions.

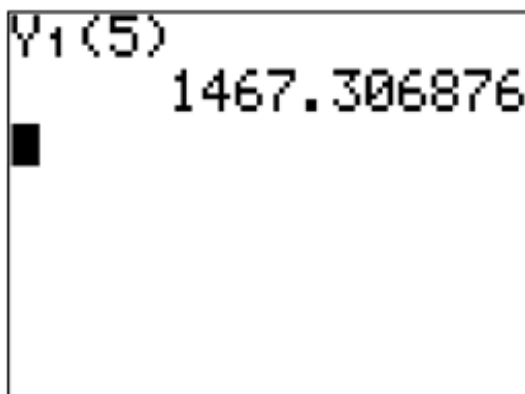
- What is the perfect amount of sleep to get before the SATs?
- Calculate the score you are predicted to get if you get 5 hours of sleep.
- What is the relevant domain of the model?
- The average SAT score is about 1500. According to the model, what amount of sleep predicts this score? Does this number represent the average number of hours that people sleep before the SATs?
- Compare the actual and predicted score for someone with 6 hours of sleep.

### Solution:

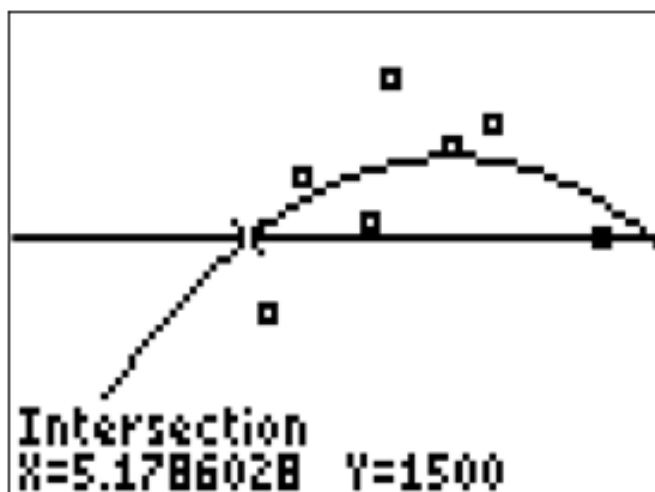
- Use the calculator to find the maximum of the parabola. The  $x$  coordinate represents the “perfect” amount of sleep.



- You can substitute  $x = 5$  into the equation, or you can let the function you created and stored in  $y_1$  simply act on the 5.



- The relevant domain is between about 2 hours and 10 hours of sleep. Beyond those numbers of sleep, the model will probably not make a whole lot of sense. How could someone get negative hours of sleep?
- You can substitute  $\hat{y} = 1500$  into the equation and solve for  $x$  using the quadratic formula, or you can graph the line  $y = 1500$  and use the calculator to produce the two intersecting points.



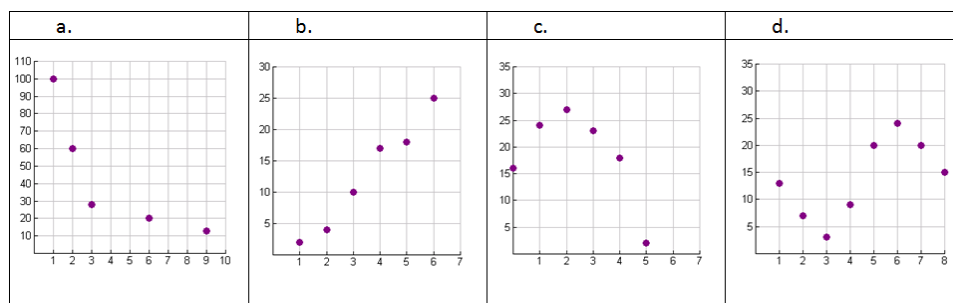
5.1786 hours and 11.7246 hours are the number hours of sleep that predict a score of 1500.

When using the model in this direction, the results do not make as much sense and you need to be extremely careful about what you say.

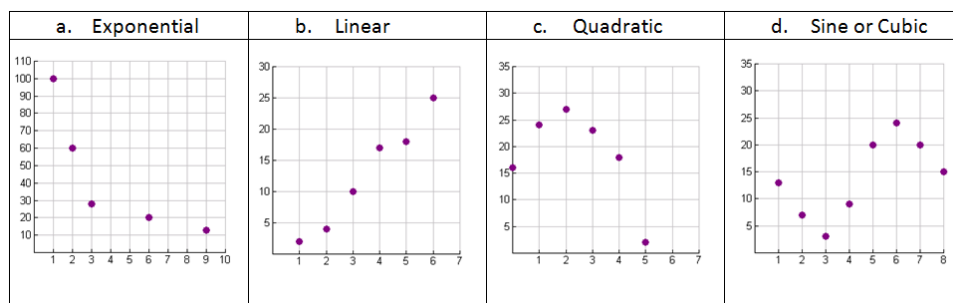
e. The actual score for someone who got 6 hours of sleep can be found in the original data to be 1720. The model predicts 1627.9970. The difference between the model and what actually happened is  $1720 - 1627.9970 = 92.0030$

### Example C

Use your knowledge of function families to predict the best model for each of the following scatterplots.

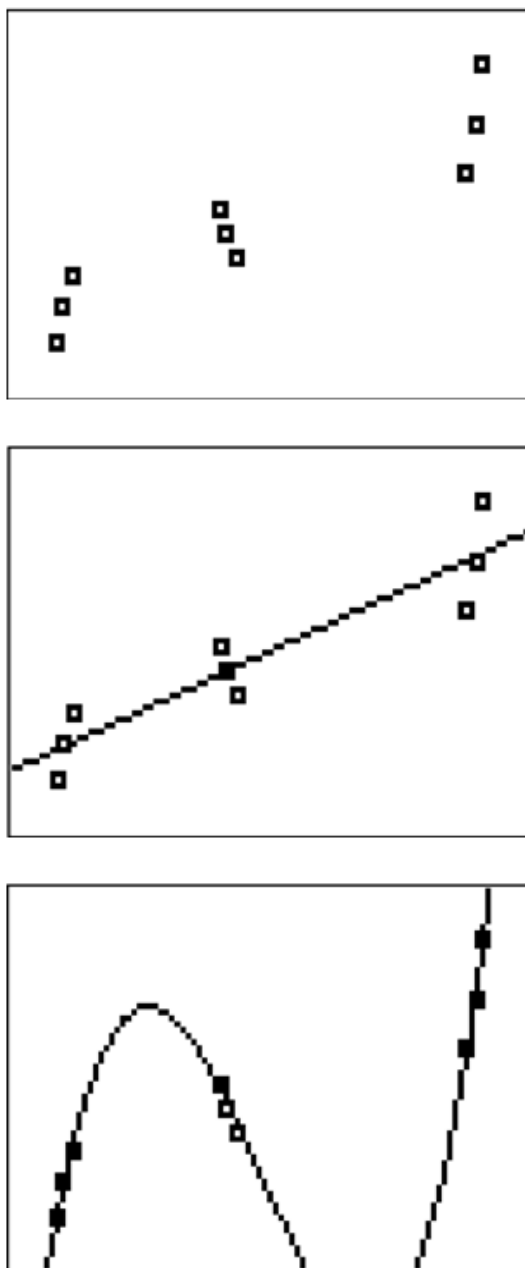


**Solution:**



### Concept Problem Revisited

For some data sets it is possible to use a polynomial or other complicated shape to exactly intersect every data point. The downside is that the model will miss the overall relationship. Consider the following data and modeling a linear relationship or cubic relationship.



The linear relationship describes the upward positive relationship in the data very well, but some points are slightly off of the line. The cubic relationship is much more accurate at the specific data points; however, there are features of the cubic relationship that differ significantly from reality when interpreted in context. In order to choose the best regression model you need to use context clues and the reasonableness of the various features of the model that fit each situation.

### Vocabulary

The *residual* is the difference between the actual height and the predicted height using the model.

### Guided Practice

1. The following data represents the height of an elephant over time. Determine the best regression function to use and determine its equation.

**TABLE 6.8:**

Age(Years)	Height (ft)
0	2
2	2.8
4	4
8	7.5
12	10
16	10.4
20	10.45

2. The following data represents the speed that Ben can kick a soccer ball at different ages. Determine the best regression function to use and determine its equation.

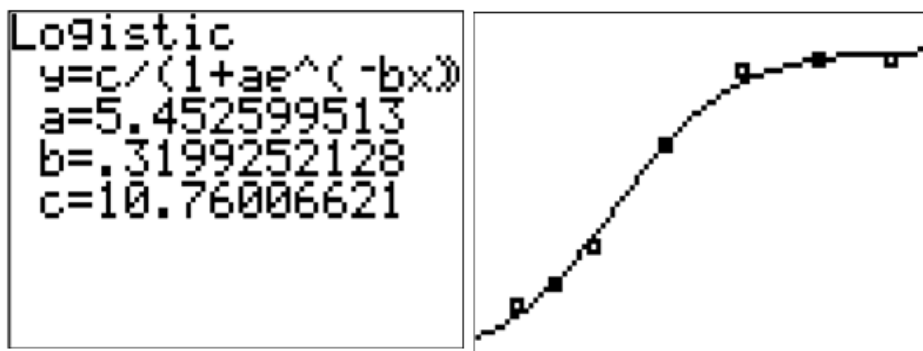
**TABLE 6.9:**

Age(Years)	Speed (mph)
4	15
10	32
20	65
30	70
50	45
60	35

3. What are two weaknesses and two strengths of the model used to predict Ben's kicking speed from Guided Practice 2?

**Answers:**

1. Logistic is the best function family because it levels off over time indicating that the elephant ceases to grow once it matures.



$$\hat{y} = \frac{10.7601}{1 + 5.4526 \cdot e^{-0.3199x}}$$

2. The best regression to use is a quadratic relationship because when Ben is little he cannot kick the ball very fast and when he is old he also cannot kick the ball very fast. Ben can kick the ball the fastest when he is an adult between the ages of 20 and 40.

$$\hat{y} = -0.05981x^2 + 4.0679x + 0.6191$$

3. One strength is that a quadratic model correctly describes the peak of kicking speed occurring in the middle of Ben's life. A linear regression might forecast Ben's kicking speed increasing forever and a logistic regression might forecast Ben's kicking speed always staying fast despite his old age. A second strength of the model could be the

y-intercept of 0.6191. Even though this number is not really in the relevant domain, it implies that as a newborn baby Ben could kick the ball very slowly which is arguably true.

One weakness of the model is that it predicts that Ben will kick the ball at 0 miles per hour at age 68.1660. This implies that Ben will not be able to kick the ball at all which isn't necessarily true.

A second weakness of the model is that it predicts negative speed at either age extreme which doesn't make sense. A better model would be flat at 0 when Ben is born and also at the end of Ben's life when he is no longer able to kick the ball.

## Practice

The table below shows the average height of an American female by age.

**TABLE 6.10:**

Age (Years)	Height (inches)
2	34
8	50
11	57
15	63
23	64
35	64

1. Determine two different equations that model the height over time using two different function families.
2. Which function is a better fit for this data? Why?
3. Use both equations to predict the y-intercepts. What does the y-intercept represent in each case? Are your predictions reasonable for this part of the graph?
4. Use your "better fit" equation to predict the height of a 70 year-old woman. Is your prediction reasonable for this part of the graph? Why or why not? What do you really need your model to do for the domain [16,100]?

Alice is in Wonderland and drinks a potion that approximately halves her height for each sip she takes, as shown in the table below.

**TABLE 6.11:**

# of sips	Height (inches)
0	60
1	29
2	16
3	8
4	4.1

5. Do an exponential regression to determine an appropriate model. What is the equation?
6. Explain why exponential regression is a good choice in this case.
7. How many sips did she take if she is 2 inches tall?
8. How tall will she be if she has 6 sips?

A rumor is spreading around your 400 person school. The following table shows the number of people who know the rumor each day.

**TABLE 6.12:**

Day	# of people who know the rumor
1	2
2	8
3	29
5	161
6	372
7	378
8	391

9. Use logistic regression to determine an equation that models the number of people who know the rumor over time.

10. Why is the logistic model appropriate in this case?

11. Use your regression equation to predict the time when only one person knew the rumor. Does this make sense?

The data table below represents how the tide changes the depth of the ocean water at a beach. At a certain place in the water, a scientist measures the depth of the water for ten consecutive hours.

**TABLE 6.13:**

Hours	Depth of Ocean Water (ft)
0	9
1	11.2
2	12.4
3	12.9
4	12.5
5	11
6	8.9
7	7
8	5.5
9	4.9
10	5.4

12. Choose the function family that is the best model for this situation and determine the regression equation.

13. Use your regression equation to predict the depth of the water at 10 hours. What is the difference between the actual depth from the data and the predicted depth from your equation (residual)?

14. Do a cubic regression on the calculator. What is the cubic regression equation? Is this a better or worse model than the model you originally chose?

15. Why might statisticians do modeling with regression for their data?

You started by working with univariate data and learned how to display it graphically and summarize it numerically. You learned how to calculate mean, median, mode and variance, and when to use each. You also explored bivariate data and used the regression capabilities of your calculator to create mathematical models for real world phenomenon.

---

## 6.9 References

1. CK-12 Foundation. . CCSA
2. CK-12 Foundation. . CCSA
3. CK-12 Foundation. . CCSA
4. CK-12 Foundation. . CCSA
5. CK-12 Foundation. . CCSA
6. CK-12 Foundation. . CCSA
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## CHAPTER

## 7

# Systems and Matrices

## Chapter Outline

---

- 7.1    SYSTEMS OF TWO EQUATIONS AND TWO UNKNOWNNS**
  - 7.2    SYSTEMS OF THREE EQUATIONS AND THREE UNKNOWNNS**
  - 7.3    MATRICES TO REPRESENT DATA**
  - 7.4    MATRIX ALGEBRA**
  - 7.5    ROW OPERATIONS AND ROW ECHELON FORMS**
  - 7.6    AUGMENTED MATRICES**
  - 7.7    DETERMINANT OF MATRICES**
  - 7.8    CRAMER'S RULE**
  - 7.9    INVERSE MATRICES**
  - 7.10   REFERENCES**
- 

You have solved systems of equations in Algebra 1 and Algebra 2 using methods such as substitution and elimination. These ideas can be extended and improved upon using matrices, which are an array of numbers.



## 7.1 Systems of Two Equations and Two Unknowns

Here you will review how to solve a system of two equations and two unknowns using the elimination method.

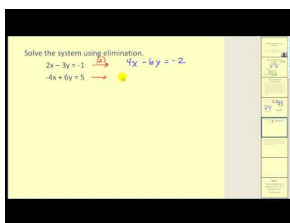
The cost of two cell phone plans can be written as a system of equations based on the number of minutes used and the base monthly rate. As a consumer, it would be useful to know when the two plans cost the same and when is one plan cheaper.

Plan A costs \$40 per month plus \$0.10 for each minute of talk time.

Plan B costs \$25 per month plus \$0.50 for each minute of talk time.

Plan B has a lower starting cost, but since it costs more per minute, it may not be the right plan for someone who likes to spend a lot of time on the phone. When do the two plans cost the same amount?

### Watch This



### MEDIA

Click image to the left or use the URL below.

URL: <http://www.ck12.org/flx/render/embeddedobject/61405>

<http://www.youtube.com/watch?v=ova8GSmPV4o> James Sousa: Solving Systems of Equations by Elimination

### Guidance

There are many ways to solve a system that you have learned in the past including substitution and graphical intersection. Here you will focus on solving using elimination because the knowledge and skills used will transfer directly into using matrices.

When solving a system, the first thing to do is to count the number of variables that are missing and the number of equations. The number of variables needs to be the same or fewer than the number of equations. Two equations and two variables can be solved, but one equation with two variables cannot.

Get into the habit of always writing systems in standard form:  $Ax + By = C$ . This will help variables line up, avoid +/- errors and lay the groundwork for using matrices. Once two equations with two variables are in standard form, decide which variable you want to eliminate, scale each equation as necessary by multiplying through by constants and then add the equations together. This procedure should reduce both the number of equations and the number of variables leaving one equation and one variable. Solve and substitute to determine the value of the second variable.

### Example A

Solve the following system of equations:  $5x + 12y = 72$  and  $3x - 2y = 18$ .

**Solution:** Here is a system of two equations and two variables in standard form. Notice that there is an  $x$  column and a  $y$  column on the left hand side and a constant column on the right hand side when you rewrite the equations as shown. Also notice that if you add the system as written no variable will be eliminated.

Equation 1:  $5x + 12y = 72$

Equation 2:  $3x - 2y = 18$

Strategically choose to eliminate  $y$  by scaling the second equation by 6 so that the coefficient of  $y$  will match at 12 and -12.

$$\begin{aligned}5x + 12y &= 72 \\18x - 12y &= 108\end{aligned}$$

Add the two equations:

$$\begin{aligned}23x &= 180 \\x &= \frac{180}{23}\end{aligned}$$

The value for  $x$  could be substituted into either of the original equations and the result could be solved for  $y$ ; however, since the value is a fraction it will be easier to repeat the elimination process in order to solve for  $x$ . This time you will take the first two equations and eliminate  $x$  by making the coefficients of  $x$  to be 15 and -15. Scale the first equation by a factor of 3 and scale the second equation by a factor of -5.

Equation 1:  $15x + 36y = 216$

Equation 2:  $-15x + 10y = -90$

Adding the two equations:

$$\begin{aligned}0x + 46y &= 126 \\y &= \frac{126}{46} = \frac{63}{23}\end{aligned}$$

The point  $(\frac{180}{23}, \frac{63}{23})$  is where these two lines intersect.

### Example B

Solve the following system of equations:

$$\begin{aligned}6x - 7y &= 8 \\15x - 14y &= 21\end{aligned}$$

**Solution:** Scaling the first equation by -2 will allow the  $y$  term to be eliminated when the equations are summed.

$$\begin{aligned}-12x + 14y &= -16 \\15x - 14y &= 21\end{aligned}$$

The sum is:

$$\begin{aligned}3x &= 5 \\x &= \frac{5}{3}\end{aligned}$$

You can substitute  $x$  into the first equation to solve for  $y$ .

$$\begin{aligned} 6 \cdot \frac{5}{3} - 7y &= 8 \\ 10 - 7y &= 8 \\ -7y &= -2 \\ y &= \frac{2}{7} \end{aligned}$$

The point  $(\frac{5}{3}, \frac{2}{7})$  is where these two lines intersect.

### Example C

Solve the following system of equations:

$$\begin{aligned} 5 \cdot \frac{1}{x} + 2 \cdot \frac{1}{y} &= 11 \\ \frac{1}{x} + \frac{1}{y} &= 4 \end{aligned}$$

**Solution:** The strategy of elimination still applies. You can eliminate the  $\frac{1}{y}$  term if the second equation is scaled by a factor of -2.

$$\begin{aligned} 5 \cdot \frac{1}{x} + 2 \cdot \frac{1}{y} &= 11 \\ -2 \cdot \frac{1}{x} - 2 \cdot \frac{1}{y} &= -8 \end{aligned}$$

Add the equations together and solve for  $x$ .

$$\begin{aligned} -3 \cdot \frac{1}{x} + 0 \cdot \frac{1}{y} &= 3 \\ -3 \cdot \frac{1}{x} &= 3 \\ \frac{1}{x} &= -1 \\ x &= -1 \end{aligned}$$

Substitute into the second equation and solve for  $y$ .

$$\begin{aligned} \frac{1}{-1} + \frac{1}{y} &= 4 \\ -1 + \frac{1}{y} &= 4 \\ \frac{1}{y} &= 5 \\ y &= \frac{1}{5} \end{aligned}$$

The point  $(-1, \frac{1}{5})$  is the point of intersection between these two curves.

### Concept Problem Revisited

Plan A costs \$40 per month plus \$0.10 for each minute of talk time.

Plan B costs \$25 per month plus \$0.50 for each minute of talk time.

If you want to find out when the two plans cost the same, you can represent each plan with an equation and solve the system of equations. Let  $y$  represent cost and  $x$  represent number of minutes.

$$y = 0.10x + 40$$

$$y = 0.50x + 25$$

First you put these equations in standard form.

$$x - 10y = -400$$

$$x - 2y = -50$$

Then you scale the second equation by -1 and add the equations together and solve for  $y$ .

$$-8y = -350$$

$$y = 43.75$$

To solve for  $x$ , you can scale the second equation by -5, add the equations together and solve for  $x$ .

$$-4x = -150$$

$$x = 37.5$$

The equivalent costs of plan A and plan B will occur at 37.5 minutes of talk time with a cost of \$43.75.

### Vocabulary

A *system of equations* is two or more equations.

*Standard form* for the equation of a line is  $Ax + By = C$ .

To *scale an equation* means to multiply every term by a constant.

### Guided Practice

1. Solve the following system using elimination:

$$20x + 6y = -32$$

$$18x - 14y = 10$$

2. Solve the following system using elimination:

$$\begin{aligned} 5x - y &= 22 \\ -2x + 7y &= 19 \end{aligned}$$

3. Solve the following system using elimination:

$$\begin{aligned} 11 \cdot \frac{1}{x} - 5 \cdot \frac{1}{y} &= -38 \\ 9 \cdot \frac{1}{x} + 2 \cdot \frac{1}{y} &= -25 \end{aligned}$$

**Answers:**

1. Start by scaling both of the equations by  $\frac{1}{2}$ . Then notice that you have  $3y$  and  $-7y$ . Rescale the first equation by 7 and the second equation by 3 to make the coefficients of  $y$  at 21 and -21. There are a number of possible ways to eliminate  $y$ .

$$\begin{aligned} 70x + 21y &= -112 \\ 27x - 21y &= 15 \end{aligned}$$

Add, solve for  $x = -1$ , substitute and solve for  $y$ .

Final Answer:  $(-1, -2)$

2. Start by scaling the first equation by 7 and notice that the  $y$  coefficient will immediately be eliminated when the equations are summed.

$$\begin{aligned} 35x - 7y &= 154 \\ -2x + 7y &= 19 \end{aligned}$$

Add, solve for  $x = \frac{173}{33}$ . Instead of substituting, practice eliminating  $x$  by scaling the first equation by 2 and the second equation by 5.

$$\begin{aligned} 10x - 2y &= 44 \\ -10x + 35y &= 95 \end{aligned}$$

Add, solve for  $y$ .

Final Answer:  $(\frac{173}{33}, \frac{139}{33})$

3. To eliminate  $\frac{1}{y}$ , scale the first equation by 2 and the second equation by 5.

To eliminate  $\frac{1}{x}$ , scale the first equation by -9 and the second equation by 11.

Final Answer:  $(-\frac{1}{3}, 1)$

**Practice**

Solve each system of equations using the elimination method.

1.  $x + y = -4$ ;  $-x + 2y = 13$

2.  $\frac{3}{2}x - \frac{1}{2}y = \frac{1}{2}$ ;  $-4x + 2y = 4$

3.  $6x + 15y = 1$ ;  $2x - y = 19$

4.  $x - \frac{2y}{3} = \frac{-2}{3}$ ;  $5x - 2y = 10$

5.  $-9x - 24y = -243$ ;  $\frac{1}{2}x + y = \frac{21}{2}$

6.  $5x + \frac{28}{3}y = \frac{176}{3}$ ;  $y + x = 10$

7.  $2x - 3y = 50$ ;  $7x + 8y = -10$

8.  $2x + 3y = 1$ ;  $2y = -3x + 14$

9.  $2x + \frac{3}{5}y = 3$ ;  $\frac{3}{2}x - y = -5$

10.  $5x = 9 - 2y$ ;  $3y = 2x - 3$

11. How do you know if a system of equations has no solution?

12. If a system of equations has no solution, what does this imply about the relationship of the curves on the graph?

13. Give an example of a system of two equations with two unknowns with an infinite number of solutions. Explain how you know the system has an infinite number of solutions.

14. Solve:

$$\begin{aligned}12 \cdot \frac{1}{x} - 18 \cdot \frac{1}{y} &= 4 \\ 8 \cdot \frac{1}{x} + 9 \cdot \frac{1}{y} &= 5\end{aligned}$$

15. Solve:

$$\begin{aligned}14 \cdot \frac{1}{x} - 5 \cdot \frac{1}{y} &= -3 \\ 7 \cdot \frac{1}{x} + 2 \cdot \frac{1}{y} &= 3\end{aligned}$$

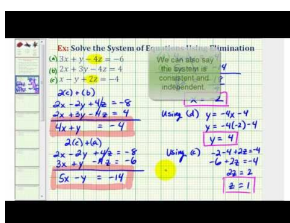
## 7.2 Systems of Three Equations and Three Unknowns

Here you will extend your knowledge of systems of equations to three equations and three unknowns.

Later, you will learn about matrices and how to row reduce which will allow you to solve systems of equations in a new way. In order to set you up so that using matrices is logical and helpful, it is important to first solve a few systems of three equations using a very specific type of variable elimination.

When solving systems, what are you allowed to do to each equation?

### Watch This



### MEDIA

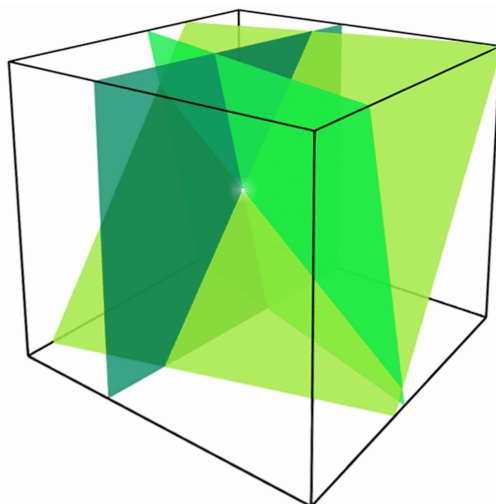
Click image to the left or use the URL below.

URL: <http://www.ck12.org/fix/render/embeddedobject/61412>

<http://www.youtube.com/watch?v=3RbVSvvRyeI> James Sousa: Ex 1: System of Three Equations with Three Unknowns Using Elimination

### Guidance

A system of three equations with three unknowns represents three planes in three dimensional space. When solving the system, you are figuring out how the planes intersect. It is possible that the three planes could intersect in a point:



It is also possible for two or more planes to be parallel or each pair of planes to intersect in a line. In either of these cases the three planes do not intersect at a single point and the system is said to have no solution. If the three planes intersect at a line or a plane, there are an infinite number of solutions.

The following system of equations has the solution (1, 3, 7). You can verify this by substituting 1 for  $x$ , 3 for  $y$ , and 7 for  $z$  into each equation.

$$\begin{aligned}x + 2y - z &= 0 \\7x - 0y + z &= 14 \\0x + y + z &= 10\end{aligned}$$

One thing to be mindful of when given a system of equations is whether or not the equations are linearly independent. Three equations are linearly independent if each equation cannot be produced by a linear combination of the other two.

When solving a system of three equations and three variables, there are a few general guidelines that can be helpful:

- Start by trying to eliminate the first variable in the second row.
- Next eliminate the first and second variables in the third row. This will create zero coefficients in the lower right hand corner.
- Repeat this process for the upper right hand corner and you should end up with a very nice diagonal indicating what  $x$ ,  $y$  and  $z$  equal.

### Example A

Even though you know the solution, solve the system of equations below:

$$\begin{aligned}x + 2y - z &= 0 \\7x - 0y + z &= 14 \\0x + y + z &= 10\end{aligned}$$

**Solution:** There are a number of ways to solve this system. Common techniques involve swapping rows, dividing and multiplying a row by a constant and adding or subtracting a multiple of one row to another.

*Step 1: Swap rows 2 and 3. Change -0 to +0.*

$$\begin{aligned}x + 2y - z &= 0 \\0x + y + z &= 10 \\7x + 0y + z &= 14\end{aligned}$$

*Step 2: Subtract 7 times row 1 to row 3.*

$$\begin{aligned}x + 2y - z &= 0 \\0x + y + z &= 10 \\0x - 14y + 8z &= 14\end{aligned}$$

*Step 3: Add 14 times row 2 to row 3*



$$\begin{aligned}x + 2y - z &= 0 \\0x + y + z &= 10 \\0x + 0y + 22z &= 154\end{aligned}$$

*Step 4: Divide row 3 by 22.*

$$\begin{aligned}x + 2y - z &= 0 \\0x + y + z &= 10 \\0x + 0y + z &= 7\end{aligned}$$

*Step 5: Subtract row 3 from row 2*

$$\begin{aligned}x + 2y - z &= 0 \\0x + y + 0z &= 3 \\0x + 0y + z &= 7\end{aligned}$$

*Step 6: Add row 3 to row 1*

$$\begin{aligned}x + 2y + 0z &= 7 \\0x + y + 0z &= 3 \\0x + 0y + z &= 7\end{aligned}$$

*Step 7: Subtract 2 times row 2 to row 1.*

$$\begin{aligned}x + 0y + 0z &= 1 \\0x + y + 0z &= 3 \\0x + 0y + z &= 7\end{aligned}$$

The solution to the system is (1, 3, 7) exactly as it should be.

### Example B

Is the following system linearly independent or dependent? How do you know?

$$\begin{aligned}3x + 2y + z &= 8 \\x + y + z &= 3 \\5x + 4y + 3z &= 14 \\6x + 6y + 6z &= 18\end{aligned}$$

**Solution:** With four equations and three unknowns there must be at least one dependent equation. The simplest method of seeing linearly dependence is to notice that one equation is just a multiple of the other. In this case the fourth equation is just six times the second equation and so it is dependent.

Most people will not notice that the third equation is also dependent. It is common to start doing a problem and notice somewhere along the way that all the variables in a row disappear. This means that the original equations were dependent. In this case, the third equation is the first equation plus two times the second equation. This means they are dependent.

### Example C

Reduce the following system to a system of two equations and two unknowns.

$$3x + 2y + z = 7$$

$$4x + 0y + z = 6$$

$$6x - y + 0z = 5$$

**Solution:** Strategically swapping rows so that the zero coefficients do not live on the diagonal is a clever starting move.

*Step 1: Swap rows 2 and 3.*

$$3x + 2y + z = 7$$

$$6x - y + 0z = 5$$

$$4x + 0y + z = 6$$

*Step 2: Scale row 3 by a factor of 3. Subtract 2 times row 1 from for 2.*

$$3x + 2y + z = 7$$

$$0x - 5y - 2z = -9$$

$$12x + 0y + 3z = 18$$

*Step 3: Subtract 4 times row 1 from row 3.*

$$3x + 2y + z = 7$$

$$0x - 5y - 2z = -9$$

$$0x - 8y - z = -10$$

*Step 4: Scale the second row by 8 and the third row by 5.*

$$3x + 2y + z = 7$$

$$0x - 40y - 16z = -72$$

$$0x - 40y - 5z = -50$$

Step 5: Subtract row 2 from row 3.

$$\begin{aligned} 3x + 2y + z &= 7 \\ 0x - 40y - 16z &= -72 \\ 0x + 0y + 11z &= +22 \end{aligned}$$

Step 6: Scale row 3 to find  $z$ .

$$\begin{aligned} 3x + 2y + z &= 7 \\ 0x - 40y - 16z &= -72 \\ 0x + 0y + z &= 2 \end{aligned}$$

Now that  $z = 2$ , rewrite the system so it becomes a system of three equations and three unknowns. Any iteration of the first two rows will work. This iteration is from step 6.

$$\begin{aligned} 3x + 2y + 2 &= 7 \\ 0x - 40y - 32 &= -72 \end{aligned}$$

### Concept Problem Revisited

When solving a system of three equations with three unknowns, you are allowed to add and subtract rows, swap rows and scale rows. These three operations should allow you to eliminate the coefficients of the variables in a systematic way.

### Vocabulary

**Swapping rows** means rewriting the system so that two rows change places. This is purely a visual reorganization and should help you problem solve and see what to do next.

**Eliminating the variable** means making the coefficient equal to zero thereby removing that variable from that particular equation and reducing the number of unknown quantities.

**Scaling a row** means multiplying every coefficient in the row by any number you choose (besides zero). This can be helpful for getting coefficients to match so that they can be eliminated.

**Consistent systems** are systems that have at least one solution. If the equations are **linearly independent**, then the system will have just one solution.

**Inconsistent systems** are systems that have no solution. An example is  $x = 2, x = 3$ .

### Guided Practice

1. When Kaitlyn went to the store with ten dollars she saw that she had some choices about what to buy. She could get one apple, one onion and one basket of blueberries for 9 dollars. She could get two apples and two onions for

10 dollars. She could also get two onions and one basket of blueberries for 10 dollars. Write a system of equations with variables  $a$ ,  $o$  and  $b$  representing each of the three things she can buy.

2. Solve the system described in # 1.
3. Show that the following system is dependent.

$$\begin{aligned}x + y + z &= 9 \\x + 2y + 3z &= 22 \\2x + 3y + 4z &= 31\end{aligned}$$

**Answers:**

1. Here is the system of equations:

$$\begin{aligned}a + o + b &= 9 \\2a + 2o &= 10 \\2o + b &= 10\end{aligned}$$

2. Rewrite the system using  $x$ ,  $y$  and  $z$  so that  $o$  and 0 do not get mixed up. Include coefficients of 0 so that each column represents one variable.

*Step 1: Rewrite*

$$\begin{aligned}1x + 1y + 1z &= 9 \\2x + 2y + 0z &= 10 \\0x + 2y + 1z &= 10\end{aligned}$$

*Step 2: Subtract 2 times row 1 from row 2.*

$$\begin{aligned}1x + 1y + 1z &= 9 \\0x + 0y - 2z &= -8 \\0x + 2y + 1z &= 10\end{aligned}$$

*Step 3: Swap row 2 and row 3. Then scale row 3.*

$$\begin{aligned}1x + 1y + 1z &= 9 \\0x + 2y + 1z &= 10 \\0x + 0y + z &= 4\end{aligned}$$

At this point you can see from the third equation that  $z = 4$ . From the second equation,  $2y + 4 = 10$ , so  $y = 3$ . Finally you can see from the first equation that  $x + 3 + 4 = 9$  so  $x = 2$ . Apples cost 2 dollars each, onions cost 3 dollars each and blueberries cost 4 dollars each.

3. You could notice that the third equation is simply the sum of the other two. What happens when you do not notice and try to solve the system as if it were independent?

*Step 1: Rewrite the system.*

$$\begin{aligned}x + y + z &= 9 \\x + 2y + 3z &= 22 \\2x + 3y + 4z &= 31\end{aligned}$$

*Step 2: Subtract 2 times row 1 from row 3.*

$$\begin{aligned}x + y + z &= 9 \\x + 2y + 3z &= 22 \\0x + 1y + 2z &= 13\end{aligned}$$

*Step 3: Subtract row 1 from row 2.*

$$\begin{aligned}x + y + z &= 9 \\0x + 1y + 2z &= 13 \\0x + 1y + 2z &= 13\end{aligned}$$

At this point when you subtract row 2 from row 3, all the coefficients in row 3 disappear. This means that you will end up with the following system of only two equations and three unknowns. Since the unknowns outnumber the equations, the system does not have a solution of one point.

$$\begin{aligned}x + y + z &= 9 \\0x + 1y + 2z &= 13\end{aligned}$$

## Practice

1. An equation with three variables represents a plane in space. Describe all the ways that three planes could interact in space.
2. What does it mean for equations to be linearly dependent?
3. How can you tell that a system is linearly dependent?
4. If you have linearly independent equations with four unknowns, how many of these equations would you need in order to get one solution?
5. Solve the following system of equations:

$$\begin{aligned}3x - 4y + z &= -17 \\6x + y - 3z &= 4 \\-x - y + 5z &= 16\end{aligned}$$

6. Show that the following system is dependent:

$$\begin{aligned}2x - 2y + z &= 5 \\6x + y - 3z &= 2 \\4x + 3y - 4z &= -3\end{aligned}$$

7. Solve the following system of equations:

$$\begin{aligned}4x + y + z &= 15 \\-2x + 3y + 4z &= 38 \\-x - y + 3z &= 16\end{aligned}$$

8. Solve the following system of equations:

$$\begin{aligned}3x - 2y + 3z &= 6 \\x + 3y - 3z &= -14 \\-x + y + 5z &= 22\end{aligned}$$

9. Solve the following system of equations:

$$\begin{aligned}3x - y + z &= -10 \\6x - 2y + 2z &= -20 \\-x - y + 4z &= 12\end{aligned}$$

10. Solve the following system of equations:

$$\begin{aligned}x - 3y + 6z &= -30 \\4x + 2y - 3z &= 18 \\-8x - 3y + 2z &= -22\end{aligned}$$

11. Solve the following system of equations.

$$\begin{aligned}x + 2y + 2z + w &= 5 \\2x + y + 2z - 0w &= 5 \\3x + 3y + 3z + 2w &= 12 \\x + 0y + z + w &= 1\end{aligned}$$

A parabola goes through (3, -9.5), (6, -32), and (-4, 8).

12. Write a system of equations that you could use to solve to find the equation of the parabola. *Hint: Use the general equation  $Ax^2 + Bx + C = y$ .*

13. Solve the system of equations from #12.

A parabola goes through  $(-2, 3)$ ,  $(2, 19)$ , and  $(1, 6)$ .

14. Write a system of equations that you could use to solve to find the equation of the parabola. *Hint: Use the general equation  $Ax^2 + Bx + C = y$ .*

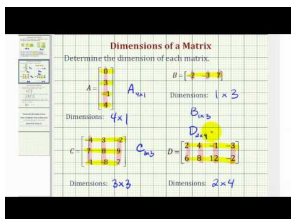
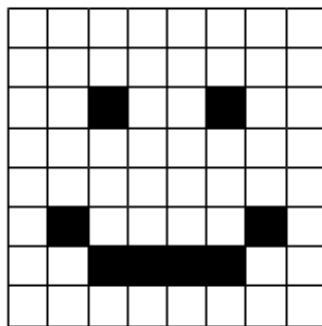
15. Solve the system of equations from #14.

## 7.3 Matrices to Represent Data

Here you will learn what a matrix is and how to use one to represent data.

A matrix is a rectangular array of numbers representing data in a variety of forms. Computers work very heavily with matrices because operations with matrices are efficient with memory. Matrices can represent statistical data with numbers, but also graphical data with pictures.

How might you use a matrix to write the following image as something a computer could recognize and work with?



### MEDIA

Click image to the left or use the URL below.

URL: <http://www.ck12.org/flx/render/embeddedobject/61418>

<http://www.youtube.com/watch?v=iIFJYjfKYjk> James Sousa: Dimensions of a Matrix

### Guidance

A matrix is a means of storing information effectively and efficiently. The rows and columns each mean something very specific and the location of a number is just as important as its value. The following are all examples of matrices:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 2 & -9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

The entries in a matrix can be written out using brackets like [ ], but they can also be described individually using a set of 2 subscript indices  $i$  and  $j$  that stand for the row number and the column number. Alternatively, the matrix can be named with just a capital letter like  $A$ .

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$



Square matrices have the same number of rows as columns. The **order** of a matrix describes the number of rows and the number of columns in the matrix. The following matrix is said to have order  $2 \times 3$  because it has two rows and three columns. A  $1 \times 1$  matrix is just a regular number.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

The identity matrix of order  $n \times n$  has zeros everywhere except along the main diagonal where it has ones. Just like the number 1 has an important property with numbers, the identity matrix of any order has special properties as well.

$$[1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

When you turn the rows of a matrix into the columns of a new matrix, the two matrices are transpositions of one another. The superscript  $T$  stands for transpose. Sometimes using the transpose of a matrix is more useful than using the matrix itself.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

A triangular matrix is not a matrix in the shape of a triangle. Rather, a lower triangular matrix is a square matrix where every entry below the diagonal is zero. An upper triangular matrix is a square matrix where every entry above the diagonal is zero. The following is a lower triangular matrix. When you work with solving matrices, look for triangular matrices because they are much easier to solve.

$$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 2 & -9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

A diagonal matrix is both upper and lower triangular which means all the entries except those along the diagonal are zero. The identity matrix is a special case of a diagonal matrix.

### Example A

Organize the driving distances between Sacramento ( $A$ ), Dallas ( $B$ ), Albany ( $C$ ) and Las Vegas ( $D$ ) in a matrix.

**Solution:**

$$\begin{array}{c} A \quad B \quad C \quad D \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} \begin{bmatrix} 0 & 1727 & 2847 & 560 \\ 1727 & 0 & 1648 & 1219 \\ 2847 & 1648 & 0 & 2552 \\ 560 & 1219 & 2552 & 0 \end{bmatrix} \end{array}$$

Note that this matrix is symmetric across the main diagonal. Symmetric matrices are important, just like triangular matrices.

### Example B

Kate runs three bakeries and each bakery sells bagels and muffins. The rows represent the bakeries and the columns represent bagels (left) and muffins (right) sold. Answer the following questions about Kate's sales.

$$K = \begin{bmatrix} 144 & 192 \\ 115 & 127 \\ 27 & 34 \end{bmatrix}$$

- What does 127 represent?
- How many muffins did Kate sell in total?
- How many bagels did Kate sell in her first location?
- Which location is doing poorly?

**Solution:**

- 127 represents the number of muffins that Kate sold in her second location. You know this because it is in the muffin column and the second row.
- The total muffins sold is equal to the sum of the right hand column.  $192 + 127 + 34 = 353$
- Kate sold 144 bagels at her first location.
- The third location is doing much worse than the other two locations.

**Example C**

Identify the order of the following matrices

$$A = \begin{bmatrix} 1 & 3 & 4 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 21 & 45 & 1 \\ 34 & 1 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 25 & 235 \\ 562 & 562 \\ 4 & 413 \\ 454 & 33 \\ 1 & 141 \end{bmatrix}$$

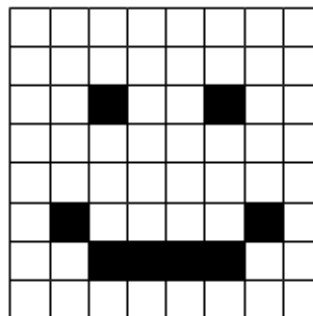
**Solution:**  $A$  is  $1 \times 4$ ,  $B$  is  $2 \times 3$ ,  $C$  is  $5 \times 2$ . Note that  $4 \times 1, 3 \times 2, 2 \times 5$  are not the same orders and would be incorrect.

**Concept Problem Revisited**

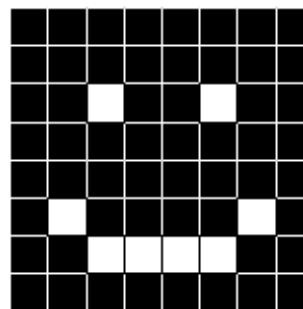
By writing every hollow square as a 0 and a blank square as a 1 a computer could read the picture:

When you use computers to manipulate images, the computer just manipulates the numbers. In this case, if you swap zeros and ones, you get the negative image.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$



Real photos and computer images have matrices that are much larger and include more numbers than just zero and one to account for more colors.

## Vocabulary

A **matrix** is a rectangular array of numbers used to represent data in a compact and efficient way.

A **square matrix** is a matrix with the same number of rows and columns.

An **identity matrix** is a matrix with zeros everywhere except along the diagonal where there are ones.

A **diagonal matrix** is a matrix with zeros everywhere except along the diagonal where the numbers can be anything.

The **transpose of a matrix** is a new matrix whose columns and rows have been switched. This changes the order of the matrix from, for example,  $3 \times 2$  to  $2 \times 3$ .

A **triangular matrix** is described as either **upper or lower triangular**. That portion of the matrix is entirely zeros. These kinds of matrices are easier to solve.

A **symmetric matrix** is a square matrix with reflection symmetry across the main diagonal.

## Guided Practice

1. Geetha took the SAT three times. The first time she scored 460 on math and 540 on verbal. The second time she scored 540 on math and 620 on verbal. The third time she scored 650 on math and 670 on verbal. Use a matrix to represent Geetha's scores.

2. Write out the  $5 \times 4$  matrix whose entries are  $a_{ij} = \frac{i+j}{j}$ .

3. Create a  $3 \times 3$  matrix for each of the following:

- Diagonal Matrix
- Lower Triangular
- Symmetric
- Identity

## Answers:

1. Let the rows represent each sitting and the columns represent math and verbal.

$$G = \begin{bmatrix} 460 & 540 \\ 540 & 620 \\ 650 & 670 \end{bmatrix}$$

2.

$$\begin{bmatrix} 2 & \frac{3}{2} & 4 & \frac{5}{4} & \frac{6}{5} \\ 3 & 2 & \frac{3}{2} & \frac{3}{2} & \frac{5}{7} \\ 4 & \frac{5}{2} & 2 & \frac{7}{4} & \frac{8}{5} \\ 5 & 3 & \frac{7}{3} & 2 & \frac{9}{5} \end{bmatrix}$$

3. While the identity matrix does technically work for all the parts of this problem, it does not highlight the differences between each definition. Here are possible answers.

a. Diagonal Matrix

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

b. Lower Triangular

$$\begin{bmatrix} 4 & 1 & 1 \\ 0 & 3 & 14 \\ 0 & 0 & 5 \end{bmatrix}$$

c. Symmetric

$$\begin{bmatrix} 4 & 1 & 1 \\ 1 & 3 & 14 \\ 1 & 14 & 5 \end{bmatrix}$$

d. Identity

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## Practice

State the order of each of the following matrices:

1.  $A = \begin{bmatrix} 4 & 2 & 4 & 7 \\ 5 & 2 & 1 & 0 \end{bmatrix}$

2.  $B = \begin{bmatrix} 0 & 1 \\ 34 & 1 \end{bmatrix}$

3.  $C = \begin{bmatrix} 2 & 62 \\ 14 & 3 \\ 4 & 3 \\ 1 & 11 \end{bmatrix}$

4.  $D = \begin{bmatrix} 12 & 0 & 2 \\ 0 & 3 & 3 \\ 4 & 0 & 1 \\ 1 & 4 & 0 \end{bmatrix}$

5.  $E = \begin{bmatrix} 1 & 11 \end{bmatrix}$

6. Give an example of a  $1 \times 1$  matrix.

7. Give an example of a  $3 \times 2$  matrix.

8. If a symmetric matrix is also lower triangular, what type of matrix is it?

9. Write out the  $2 \times 3$  matrix whose entries are  $a_{ij} = i - j$ .

Morgan worked for three weeks during the summer earning money on Mondays, Tuesdays, Wednesdays, Thursdays, and Fridays. The following matrix represents his earnings.

$$\begin{bmatrix} 24 & 22 & 32 \\ 25 & 28 & 30 \\ 30 & 28 & 32 \\ 10 & 15 & 19 \\ 35 & 32 & 30 \end{bmatrix}$$

10. What do the rows and columns represent?
11. How much money did Morgan make in the first week?
12. How much money did Morgan make on Tuesdays?
13. What day of the week was most profitable?
14. What day of the week was least profitable?
15. Is the following a matrix? Explain.

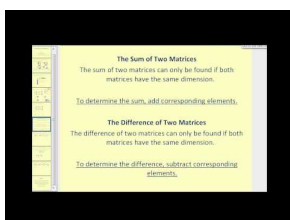
$$\begin{bmatrix} \text{dogs} & 0 \\ \text{cats} & 3 \\ \text{sheep} & 0 \\ \text{ducks} & 4 \end{bmatrix}$$

## 7.4 Matrix Algebra

Here you will add, subtract and multiply matrices. As a result you will discover the algebraic properties of matrices.

Algebra refers to your ability to manipulate variables and unknowns based on rules and properties. Matrix algebra is extremely similar to the algebra you already know for numbers with a few important differences. What are these differences?

### Watch This

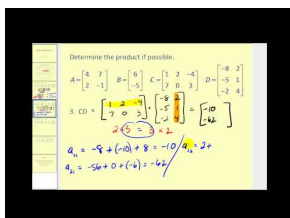


#### MEDIA

Click image to the left or use the URL below.

URL: <http://www.ck12.org/flx/render/embeddedobject/61448>

<http://www.youtube.com/watch?v=iNty4CSFIpU> James Sousa: Matrix Addition, Subtraction, and Scalar Multiplication

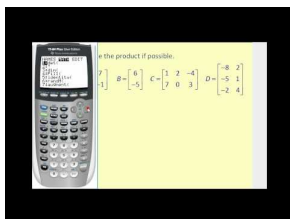


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Click image to the left or use the URL below.

URL: <http://www.ck12.org/flx/render/embeddedobject/61450>

<http://www.youtube.com/watch?v=6Hmzu-WKCjc> James Sousa: Matrix Multiplication



#### MEDIA

Click image to the left or use the URL below.

URL: <http://www.ck12.org/flx/render/embeddedobject/61452>

<http://www.youtube.com/watch?v=C7Dc414qmlk> James Sousa: Matrix Multiplication on the Calculator

### Guidance

Two matrices of the same order can be added by summing the entries in the corresponding positions.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 7 & 7 \\ 7 & 7 & 7 \end{bmatrix}$$

Two matrices of the same order can be subtracted by subtracting the entries in the corresponding positions.

$$\begin{bmatrix} 10 & 9 & 8 \\ 7 & 6 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 6 \\ 5 & 4 & 3 \end{bmatrix}$$

You can find the product of matrix  $A$  and matrix  $B$  if the number of columns in matrix  $A$  matches the number of rows in matrix  $B$ . Another way to remember this is when you write the orders of matrix  $A$  and matrix  $B$  next to each other they must be connected by the same number. The resulting matrix has the number of rows from the first matrix and the number of columns from the second matrix.

$$(2 \times 3) \cdot (3 \times 5) = (2 \times 5)$$

To compute the first entry of the resulting  $2 \times 5$  matrix you should match the first row from the first matrix and the first column of the second matrix. The arithmetic operation to combine these numbers is identical to taking the dot product between two vectors.

$$\begin{bmatrix} 1 & 4 & 3 \\ 5 & 6 & 9 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 3 & 1 & 0 \\ 2 & 0 & 0 & 2 & 1 \\ 1 & 1 & 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & ? & ? & ? & ? \\ 21 & ? & ? & ? & ? \end{bmatrix}$$

- The entry in the first row first column of the new matrix is computed as  $1 \cdot 0 + 4 \cdot 2 + 3 \cdot 1 = 11$ .
- The entry in the second row first column of the new matrix is computed as  $5 \cdot 0 + 6 \cdot 2 + 9 \cdot 1 = 21$ .
- The rest of the entries of this product are left to Example A.

### Properties of Matrix Algebra

- Commutativity holds for matrix addition. This means that when matrices  $A$  and  $B$  can be added (when they have matching orders), then:  $A + B = B + A$
- Commutativity **does not** hold in general for matrix multiplication.
- Associativity holds for both multiplication and addition.  $(AB)C = A(BC)$ ,  $(A + B) + C = A + (B + C)$
- Distribution over addition and subtraction holds.  $A(B \pm C) = AB \pm AC$

### Example A

Complete the entries of the matrix multiplication introduced in the guidance section.

$$\begin{bmatrix} 1 & 4 & 3 \\ 5 & 6 & 9 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 3 & 1 & 0 \\ 2 & 0 & 0 & 2 & 1 \\ 1 & 1 & 3 & 0 & 1 \end{bmatrix}$$

**Solution:** Two of the arithmetic operations are shown.

$$C = \begin{bmatrix} 11 & 4 & 12 & 9 & 7 \\ 21 & 14 & 42 & 17 & 15 \end{bmatrix}$$

$$c_{12} = 1 \cdot 1 + 4 \cdot 0 + 3 \cdot 1 = 4$$

$$c_{22} = 5 \cdot 1 + 6 \cdot 0 + 9 \cdot 1 = 14$$

### Example B

Show the commutative property does not hold by demonstrating  $AB \neq BA$

$$A = \begin{bmatrix} 0 & -1 & 8 \\ 1 & 2 & 0 \\ 4 & 3 & 12 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 0 \end{bmatrix}$$

**Solution:**

$$AB = \begin{bmatrix} 30 & 22 & -1 \\ 5 & 9 & 3 \\ 58 & 62 & 7 \end{bmatrix}$$

$$BA = \begin{bmatrix} 9 & 12 & 20 \\ 6 & 5 & 28 \\ 3 & 2 & 32 \end{bmatrix}$$

### Example C

Compute the following matrix arithmetic:  $10 \cdot (2A - 3C) \cdot B$ .

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 2 \\ 4 & 3 & 2 \end{bmatrix}, C = \begin{bmatrix} 12 & 0 \\ 1 & 3 \end{bmatrix}$$

**Solution:** When a matrix is multiplied by a scalar (such as with  $2A$ ), multiply each entry in the matrix by the scalar.

$$2A = \begin{bmatrix} 2 & 4 \\ 8 & 10 \end{bmatrix}$$

$$-3C = \begin{bmatrix} -36 & 0 \\ -3 & -9 \end{bmatrix}$$

$$2A - 3C = \begin{bmatrix} -34 & 4 \\ 5 & 1 \end{bmatrix}$$

Since the associative property holds, you can either distribute the ten or multiply by matrix  $B$  next.

$$(2A - 3C) \cdot B = \begin{bmatrix} 16 & -22 & -60 \\ 4 & 8 & 12 \end{bmatrix}$$

$$10 \cdot (2A - 3C) \cdot B = \begin{bmatrix} 160 & -220 & -600 \\ 40 & 80 & 120 \end{bmatrix}$$

### Concept Problem Revisited

The main difference between matrix algebra and regular algebra with numbers is that matrices do not have the commutative property for multiplication. There are other complexities that matrices have, but many of them stem from the fact that for most matrices  $AB \neq BA$ .

### Vocabulary

**Matrix operations** are addition, subtraction and multiplication. Division involves a multiplicative inverse that is not discussed at this point.



**Guided Practice**

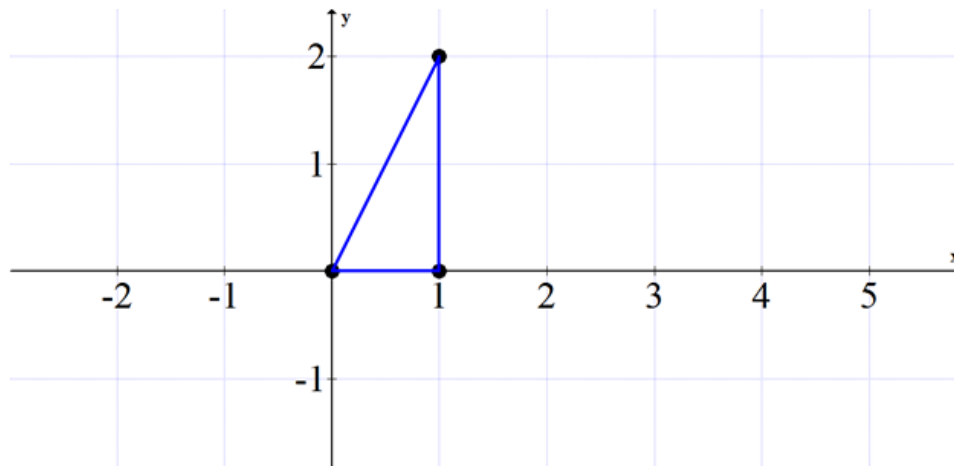
1. Show that a  $3 \times 3$  identity matrix works as the multiplicative identity.
2. Use your calculator to input and compute the following matrix operations.

$$A = \begin{bmatrix} 54 & 65 & 12 \\ 235 & 322 & 167 \\ 413 & 512 & 123 \end{bmatrix}, \quad B = \begin{bmatrix} 163 & 212 & 466 \\ 91 & 221 & 184 \\ 42 & 55 & 42 \end{bmatrix}$$

$$A^T \cdot B \cdot A - 100A$$

3. Matrix multiplication can be used as a transformation in the coordinate system. Consider the triangle with coordinates  $(0, 0)$   $(1, 2)$  and  $(1, 0)$  the following matrix:

$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{bmatrix}$$



What does the new picture look like?

**Answers:**

1. A  $3 \times 3$  matrix multiplied by the identity should yield the original matrix.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$a_{11} = a \cdot 1 + b \cdot 0 + c \cdot 0 = a$$

$$a_{12} = a \cdot 0 + b \cdot 1 + c \cdot 0 = b$$

$$\vdots$$

2. Most graphing calculators like the TI-84 can do operations on matrices. Find where you can enter matrices and enter the two matrices.

$[A] = \begin{bmatrix} 54 & 65 & 12 \\ 235 & 322 & 167 \\ 413 & 512 & 123 \end{bmatrix}$	$[B] = \begin{bmatrix} 163 & 212 & 466 \\ 91 & 221 & 184 \\ 42 & 55 & 42 \end{bmatrix}$
--	---

Then type in the appropriate operation and see the result. The TI-84 has a built in Transpose button.

$$[A] \cdot [B]^T - 100 = \begin{bmatrix} 58209162 & 747107 \\ 75731470 & 97231 \\ 29158404 & 375219 \end{bmatrix}$$

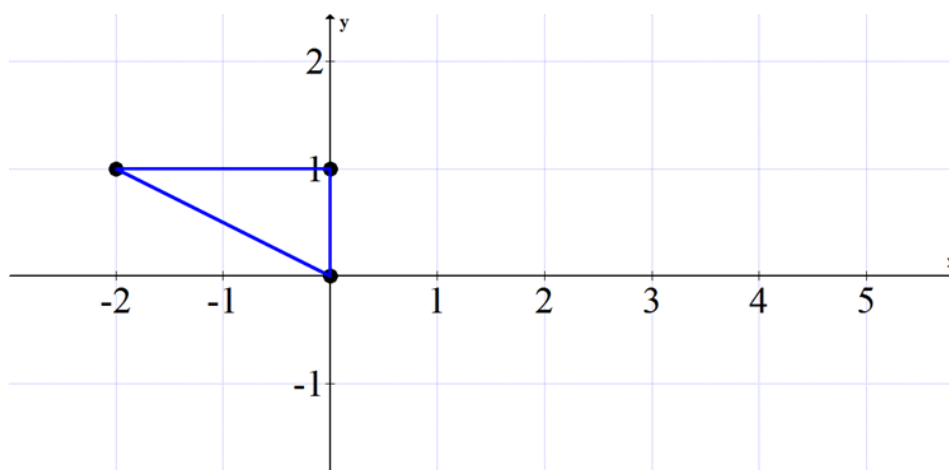
The actual numbers on this guided practice are less important than the knowledge that your calculator can perform all of the matrix algebra demonstrated in this concept. It is useful to fully know the capabilities of the tools at your disposal, but it should not replace knowing why the calculator does what it does.

3. The matrix simplifies to become:

$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

When applied to each point as a transformation, a new point is produced. Note that  $\begin{bmatrix} x & y \end{bmatrix}$  is a matrix representing each original point and  $\begin{bmatrix} x' & y' \end{bmatrix}$  is the new point. The  $x'$  is read as “x prime” and is a common way to refer to a result after a transformation.

$$\begin{aligned} \begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} &= \begin{bmatrix} x' & y' \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} &= \begin{bmatrix} -2 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \end{bmatrix} \end{aligned}$$



Notice how the matrix transformation rotates graphs in a counterclockwise direction  $90^\circ$ .

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} -y & x \end{bmatrix}$$

The matrix transformation applied in the following order will rotate a graph clockwise  $90^\circ$ .

$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}$$

### Practice

Do #1-#11 without your calculator.

$$A = \begin{bmatrix} 2 & 7 \\ 3 & 8 \end{bmatrix}, B = \begin{bmatrix} 0 & 5 & 1 \\ 3 & 4 & 6 \end{bmatrix}, C = \begin{bmatrix} 14 & 6 \\ 1 & 2 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 \\ 1 & 2 \end{bmatrix}$$

1. Find  $AC$ . If not possible, explain.
2. Find  $BA$ . If not possible, explain.
3. Find  $CA$ . If not possible, explain.
4. Find  $4B^T$ . If not possible, explain.
5. Find  $A + C$ . If not possible, explain.
6. Find  $D - A$ . If not possible, explain.
7. Find  $2(A + C - D)$ . If not possible, explain.
8. Find  $(A + C)B$ . If not possible, explain.
9. Find  $B(A + C)$ . If not possible, explain.
10. Show that  $A(C + D) = AC + AD$ .
11. Show that  $A(C - D) = AC - AD$ .

Practice using your calculator for #12-#15.

$$E = \begin{bmatrix} 312 & 59 & 34 \\ 342 & 156 & 189 \\ 783 & 23 & 133 \end{bmatrix}, F = \begin{bmatrix} 33 & 72 & 21 \\ 93 & 41 & 94 \\ 62 & 75 & 72 \end{bmatrix}, G = \begin{bmatrix} 11 & 735 & 67 \\ 93 & 456 & 2 \\ 94 & 34 & 0 \end{bmatrix}$$

12. Find  $E + F + G$ .
13. Find  $2E$ .

14. Find  $4F$ .

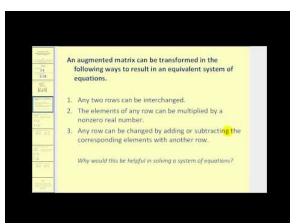
15. Find  $(E + F)G$ .

## 7.5 Row Operations and Row Echelon Forms

Here you will manipulate matrices using row operations into row echelon form and reduced row echelon form.

Applying row operations to reduce a matrix is a procedural skill that takes lots of writing, rewriting and careful arithmetic. The payoff for being able to transform a matrix into a simplified form will become clear later. For now, what does the simplified form mean for a matrix?

### Watch This



### MEDIA

Click image to the left or use the URL below.

URL: <http://www.ck12.org/flx/render/embeddedobject/61454>

<http://www.youtube.com/watch?v=LsnOINjWWug> James Sousa: Introduction to Augmented Matrices

### Guidance

There are only three operations that are permitted to act on matrices. They are the exact same operations that are permitted when solving a system of equations.

1. Add a multiple of one row to another row.
2. Scale a row by multiplying through by a non-zero constant.
3. Swap two rows.

Using these three operations, your job is to simplify matrices into **row echelon form**. Row echelon form must meet three requirements.

1. The leading coefficient of each row must be a one.
2. All entries in a column below a leading one must be zero.
3. All rows that just contain zeros are at the bottom of the matrix.

Here are some examples of matrices in row echelon form:

$$\begin{bmatrix} 1 & 14 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 5 & 6 \\ 0 & 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**Reduced row echelon form** also has one extra stipulation compared with row echelon form.

4. Every leading coefficient of 1 must be the only non-zero element in that column.

Here are some examples of matrices in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Putting a matrix into reduced row echelon form is a result of performing **Gauss-Jordan elimination**. The process illustrated in this concept is named after those mathematicians.

### Example A

Put the following matrix into reduced row echelon form.

$$\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$$

**Solution:** In each step of the solution, only one of the three row operations will be used. Specific shorthand will be introduced.

$$\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 7 \\ 6 & 15 \end{bmatrix} \rightarrow -2 \cdot I \rightarrow \begin{bmatrix} 3 & 7 \\ 0 & 1 \end{bmatrix}$$

Note that the  $\cdot 3$  in between the first two matrices indicates that the second row is scaled by a factor of 3. The  $-2 \cdot I$  between the next two matrices indicates that the second row has two times the first row subtracted from it. The  $I$  is a roman numeral referring to the row number.

$$\begin{bmatrix} 3 & 7 \\ 0 & 1 \end{bmatrix} \rightarrow -7III \rightarrow \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \cdot \frac{1}{3} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Row reducing a  $2 \times 2$  matrix to become the identity matrix is an exercise that illustrates the fact that the rows were linearly independent.

### Example B

Put the following matrix into reduced row echelon form.

$$\begin{bmatrix} 2 & 4 & 0 \\ 0 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

**Solution:**

$$\begin{bmatrix} 2 & 4 & 0 \\ 0 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} \rightarrow -\frac{I}{2} \rightarrow \begin{bmatrix} 2 & 4 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix} \rightarrow \div 4 \rightarrow \begin{bmatrix} 2 & 4 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \div 2 \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \div 3 \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

Note that in the preceding step, two operations were used. This is acceptable when the operations do not interfere or interact with each other.

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} \rightarrow -\frac{III}{3} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow -2II \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Again, row reducing a  $3 \times 3$  matrix to become the identity matrix is just an exercise that illustrates the fact that the rows were linearly independent.

### Example C

In a single  $3 \times 3$  matrix, describe the general approach of Gauss-Jordan elimination. In other words, which locations would you try to focus on first?

**Solution:** One approach is to try to get a one in the A position. Then get a zero in position B and position C by multiplying by a multiple of row 1. Then try to get a zero in position D.

$$\begin{bmatrix} A & I & G \\ B & H & F \\ C & D & E \end{bmatrix}$$

Every matrix may have a different strategy and as long as you use the three row operations, you will be on the right track. One thing to be very careful of is to try to avoid fractions within your matrix. Scale the row to eliminate the fraction.

### Concept Problem Revisited

There are two forms of a matrix that are most simplified. The most important is reduced row echelon form that follows the four stipulations from the guidance section. An example of a matrix in reduced row echelon form is:

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 43 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 98 & 5 \end{bmatrix}$$

### Vocabulary

**Row operations** are swapping rows, adding a multiple of one row to another or scaling a row by multiplying through by a scalar.

**Row echelon form** is a matrix that has a leading one at the start of every non-zero row, zeros below every leading one and all rows containing only zeros at the bottom of the matrix.

**Reduced row echelon form** is the same as row echelon form with one additional stipulation: that every other entry in a column with a leading one must be zero.

### Guided Practice

1. Reduce the following matrix to reduced row echelon form.

$$\begin{bmatrix} 0 & 4 & 5 \\ 2 & 6 & 8 \end{bmatrix}$$

2. Reduce the following matrix to row echelon form.

$$\begin{bmatrix} 3 & 6 \\ 2 & 4 \\ 5 & 17 \end{bmatrix}$$

3. Reduce the following matrix to reduced row echelon form.

$$\begin{bmatrix} 3 & 4 & 1 & 0 \\ 5 & -1 & 0 & 1 \end{bmatrix}$$

### Answers:

1.

$$\begin{bmatrix} 0 & 4 & 5 \\ 2 & 6 & 8 \end{bmatrix} \rightarrow II \rightarrow \begin{bmatrix} 2 & 6 & 8 \\ 0 & 4 & 5 \end{bmatrix} \rightarrow \div 2 \rightarrow \begin{bmatrix} 1 & 3 & 4 \\ 0 & 4 & 5 \end{bmatrix}$$

$$\rightarrow -I \rightarrow \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow -3II \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$2. \begin{bmatrix} 3 & 6 \\ 2 & 4 \\ 5 & 17 \end{bmatrix} \rightarrow \div 3 \rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 5 & 17 \end{bmatrix} \rightarrow -I \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 7 \end{bmatrix} \rightarrow III \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 7 \\ 0 & 0 \end{bmatrix} \rightarrow \div 7 \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

3.

$$\begin{aligned} \begin{bmatrix} 3 & 4 & 1 & 0 \\ 5 & -1 & 0 & 1 \end{bmatrix} &\rightarrow \cdot 5 \rightarrow \begin{bmatrix} 15 & 20 & 5 & 0 \\ & 15 & -3 & 0 \end{bmatrix} \rightarrow -I \rightarrow \begin{bmatrix} 15 & 20 & 5 & 0 \\ 0 & -23 & -5 & 3 \end{bmatrix} \\ &\rightarrow \cdot 23 \rightarrow \begin{bmatrix} 345 & 460 & 115 & 0 \\ 0 & -460 & -115 & 60 \end{bmatrix} \\ &\rightarrow \cdot 20 \rightarrow \begin{bmatrix} 345 & 460 & 115 & 0 \\ 0 & -460 & -115 & 60 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} 345 & 460 & 115 & 0 \\ 0 & -460 & -115 & 60 \end{bmatrix} &\rightarrow +II \rightarrow \begin{bmatrix} 345 & 0 & 0 & 60 \\ 0 & -460 & -115 & 60 \end{bmatrix} \\ &\rightarrow \div 345 \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{60}{345} \\ 0 & 1 & \frac{115}{460} & -\frac{60}{460} \end{bmatrix} \\ &\rightarrow \div -460 \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{60}{345} \\ 0 & 1 & \frac{115}{460} & -\frac{60}{460} \end{bmatrix} \end{aligned}$$

Notice how fractions were avoided until the final step. Adding and subtracting large numbers in a matrix is easier to handle than adding and subtracting small numbers because then you don't need to find a common denominator.

### Practice

1. Give an example of a matrix in row echelon form.
2. Give an example of a matrix in reduced row echelon form.
3. What are the three row operations you are allowed to perform when reducing a matrix?
4. If a square matrix reduces to the identity matrix, what does that mean about the rows of the original matrix?

Use the following matrix for 5-6.

$$A = \begin{bmatrix} -3 & -4 & -12 \\ 4 & 4 & 12 \\ -11 & -12 & -35 \end{bmatrix}$$

5. Reduce matrix  $A$  to row echelon form.
6. Reduce matrix  $A$  to reduced row echelon form. Are the rows of matrix  $A$  linearly independent?

Use the following matrix for 7-8.

$$B = \begin{bmatrix} 3 & -4 & 8 \\ 9 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

7. Reduce matrix  $B$  to row echelon form.
8. Reduce matrix  $B$  to reduced row echelon form. Are the rows of matrix  $B$  linearly independent?

Use the following matrix for 9-10.

$$C = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 3 & 6 & -3 & 1 \\ 6 & 12 & -7 & 0 \end{bmatrix}$$

9. Reduce matrix  $C$  to row echelon form.
10. Reduce matrix  $C$  to reduced row echelon form. Are the rows of matrix  $C$  linearly independent?

Use the following matrix for 11-12.

$$D = \begin{bmatrix} 1 & 1 \\ 3 & 4 \\ 2 & 3 \end{bmatrix}$$

11. Reduce matrix  $D$  to row echelon form.



12. Reduce matrix  $D$  to reduced row echelon form. Are the rows of matrix  $D$  linearly independent?

Use the following matrix for 13-14.

$$E = \begin{bmatrix} -5 & -6 & -12 \\ -1 & -1 & -2 \\ 2 & 2 & 4 \end{bmatrix}$$

13. Reduce matrix  $E$  to row echelon form.

14. Reduce matrix  $E$  to reduced row echelon form. Are the rows of matrix  $E$  linearly independent?

Use the following matrix for 15-16.

$$F = \begin{bmatrix} -23 & 6 & 3 \\ 2 & -\frac{1}{2} & 0 \\ -8 & 2 & 1 \end{bmatrix}$$

15. Reduce matrix  $F$  to row echelon form.

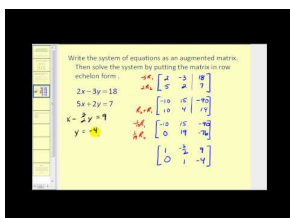
16. Reduce matrix  $F$  to reduced row echelon form. Are the rows of matrix  $F$  linearly independent?

## 7.6 Augmented Matrices

Here you will solve systems of equations using augmented matrices.

The reason why the rules for row reducing matrices are the same as the rules for eliminating coefficients when solving a system of equations is because you are essentially doing the same thing in each case. When you write and rewrite the equation every time you end up writing down lots of extra information. Matrices take care of this information by embedding it in the location of each entry. How would you use matrices to write the following system of equations?

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URL: <http://www.ck12.org/flx/render/embeddedobject/61456>

<http://www.youtube.com/watch?v=BWBckWPjfpw> James Sousa: Augmented Matrices: Row Echelon Form

### Guidance

In order to represent a system as a matrix equation, first write all the equations in standard form so that the coefficients of the variables line up in columns. Then copy down just the coefficients in a matrix array. Next copy the variables in a variable matrix and the constants into a constant matrix.

$$\begin{aligned}x + y + z &= 9 \\x + 2y + 3z &= 22 \\2x + 3y + 4z &= 31\end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 22 \\ 31 \end{bmatrix}$$

The reason why this works is because of the way matrix multiplication is defined.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1x + 1y + 1z \\ 1x + 2y + 3z \\ 2x + 3y + 4z \end{bmatrix} = \begin{bmatrix} 9 \\ 22 \\ 31 \end{bmatrix}$$

Notice how putting brackets around the two matrices on the right does very little to hide the fact that this is just a regular system of 3 equations and 3 variables.

Once you have your system represented as a matrix you can solve it using an augmented matrix. An augmented matrix is two matrices that are joined together and operated on as if they were a single matrix. In the case of solving a system, you need to augment the coefficient matrix and the constant matrix. The vertical line indicates the separation between the coefficient matrix and the constant matrix.

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 1 & 2 & 3 & 22 \\ 2 & 3 & 4 & 31 \end{array} \right]$$

Reduce the matrix to reduced row echelon form and you will find the solution to the system, if one exists.

### Example A

Attempt to solve the system from the guidance section.

**Solution:**

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 1 & 2 & 3 & 22 \\ 2 & 3 & 4 & 31 \end{array} \right] \rightarrow \begin{array}{l} \rightarrow -I \\ \rightarrow -2I \end{array} \rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & 1 & 2 & 13 \end{array} \right] \rightarrow \begin{array}{l} \rightarrow -II \\ \rightarrow -II \end{array} \rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This system is dependent which means there are an infinite number of solutions.

### Example B

Solve the following system using an augmented matrix.

$$\begin{aligned} x + y + z &= 6 \\ x - y - z &= -4 \\ x + 2y + 3z &= 14 \end{aligned}$$

**Solution:**

$$\begin{aligned} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & -1 & -1 & -4 \\ 1 & 2 & 3 & 14 \end{array} \right] &\rightarrow \begin{array}{l} \rightarrow -I \\ \rightarrow -I \end{array} \rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & -2 & -10 \\ 0 & 1 & 2 & 8 \end{array} \right] \rightarrow \begin{array}{l} \rightarrow -III \\ \rightarrow +3III \end{array} \rightarrow \\ \left[ \begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 4 & 14 \\ 0 & 1 & 2 & 8 \end{array} \right] &\rightarrow \begin{array}{l} \rightarrow -II \\ \rightarrow -II \end{array} \rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 4 & 14 \\ 0 & 0 & -2 & -6 \end{array} \right] \rightarrow \begin{array}{l} \rightarrow \div -2 \\ \rightarrow \div -2 \end{array} \rightarrow \\ \left[ \begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 4 & 14 \\ 0 & 0 & 1 & 3 \end{array} \right] &\rightarrow \begin{array}{l} \rightarrow +III \\ \rightarrow -4III \end{array} \rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right] \end{aligned}$$

Every matrix can be interpreted as its own linear system. The final augmented matrix can be interpreted as:

$$\begin{aligned} 1x + 0y + 0z &= 1 \\ 0x + 1y + 0z &= 2 \\ 0x + 0y + 1z &= 3 \end{aligned}$$

Which means  $x = 1, y = 2, z = 3$ .

### Example C

Solve the following system using augmented Matrices.

$$w + x + z = 11$$

$$w + x = 9$$

$$x + y = 7$$

$$y + z = 5$$

**Solution:** While substitution would work in this problem, the idea is to demonstrate how augmented matrices will work even with larger matrices.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 11 \\ 1 & 1 & 0 & 0 & 9 \\ 0 & 1 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 & 5 \end{array}\right] \begin{array}{l} \rightarrow \\ \rightarrow IV \\ \rightarrow II \\ \rightarrow III \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 11 \\ 0 & 1 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 & 5 \\ 1 & 1 & 0 & 0 & 9 \end{array}\right] \begin{array}{l} \rightarrow \\ \rightarrow \\ \rightarrow \\ \rightarrow -I \end{array} \rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 11 \\ 0 & 1 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & -1 & -2 \end{array}\right] \begin{array}{l} \rightarrow \\ \rightarrow \\ \rightarrow \\ \rightarrow \cdot(-1) \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 11 \\ 0 & 1 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} \rightarrow -IV \\ \rightarrow \\ \rightarrow -IV \\ \rightarrow \end{array} \rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 9 \\ 0 & 1 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} \rightarrow \\ \rightarrow -III \\ \rightarrow \\ \rightarrow \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} \rightarrow -II \\ \rightarrow \\ \rightarrow \\ \rightarrow \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$$w = 5, x = 4, y = 3, z = 2$$

### Concept Problem Revisited

If you were to write the system as a matrix system you could write:

$$5x + y = 6$$

$$x + y = 10$$

$$\begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

### Vocabulary

An **augmented matrix** is a matrix formed when two matrices are joined together and operated on as if they were a single matrix.

### Guided Practice

1. Use an augmented matrix to solve the following system.

$$2x + y + z = 16$$

$$2y + 6z = 0$$

$$x + y = 10$$

2. Use an augmented matrix to solve the following system.

$$3x + y = -15$$

$$x + 2y = 15$$

3. Use an augmented matrix to solve the following system.

$$-a + b - c = 0$$

$$2a - 2b - 3c = 25$$

$$3a - 4b + 3c = 2$$

### Answers:

1. The row reduction steps are not shown. Only the initial and final augmented matrices are shown.

$$\left[ \begin{array}{ccc|c} 2 & 1 & 1 & 16 \\ 0 & 2 & 6 & 0 \\ 1 & 1 & 0 & 10 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

$$2. \left[ \begin{array}{cc|c} 3 & 1 & -15 \\ 1 & 2 & 15 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & 0 & -9 \\ 0 & 1 & 12 \end{array} \right]$$

$$3. \left[ \begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 2 & -2 & -3 & 25 \\ 3 & -4 & 3 & 2 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -5 \end{array} \right]$$

### Practice

Solve the following systems of equations using augmented matrices. If one solution does not exist, explain why not.

1.

$$4x - 2y = -20$$

$$x - 3y = -15$$

2.

$$3x + 5y = 33$$

$$-x - 2y = -13$$

3.

$$x + 4y = 11$$

$$3x + 12y = 33$$

4.

5.

$$\begin{aligned}-3x + y &= -7 \\ -x + 4y &= 5\end{aligned}$$

6.

$$\begin{aligned}3x + y &= 6 \\ -6x - 2y &= 10\end{aligned}$$

7.

$$\begin{aligned}2x - y + z &= 4 \\ 4x + 7y - z &= 38 \\ -x + 3y + 2z &= 23\end{aligned}$$

8.

$$\begin{aligned}4x + y - z &= -16 \\ -3x + 4y + z &= 18 \\ x + y - 3z &= -17\end{aligned}$$

9.

$$\begin{aligned}3x + 2y - 3z &= 7 \\ -x + 5y + 2z &= 29 \\ x + 2y + z &= 15\end{aligned}$$

10.

$$\begin{aligned}2x + y - 2z &= 4 \\ -4x - 2y + 4z &= -8 \\ 3x + y - z &= 5\end{aligned}$$

11.

$$\begin{aligned}-x + 3y + z &= 11 \\ 3x + y + 2z &= 27 \\ 5x - y - z &= 5\end{aligned}$$

$$\begin{aligned}3x + 2y + 4z &= 21 \\ -2x + 3y + z &= -11 \\ x + 2y - 3z &= -3\end{aligned}$$

12.

$$\begin{aligned}-x + 2y - 6z &= 4 \\ 8x + 5y + 3z &= -8 \\ 2x - 4y + 12z &= 5\end{aligned}$$

13.

$$\begin{aligned}3x + 5y + 8z &= 37 \\ -6x + 3y + z &= 42 \\ x + 3y - 2z &= 5\end{aligned}$$

14.

$$\begin{aligned}4x + y - 6z &= -38 \\ 2x + 7y + 8z &= 108 \\ -3x + 2y - 3z &= -15\end{aligned}$$

15.

$$\begin{aligned}6x + 3y - 2z &= -22 \\ -4x - 2y + 4z &= 28 \\ 3x + 3y + 2z &= 7\end{aligned}$$

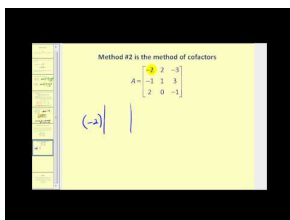
## 7.7 Determinant of Matrices

Here you will find the determinants of  $2 \times 2$  and higher order matrices.

A **determinant** is a number computed from the entries in a square matrix. It has many properties and interpretations that you will explore in linear algebra. This concept is focused on the procedure of calculating determinants. Once you know how to calculate the determinant of a  $2 \times 2$  matrix, then you will be able to calculate the determinant of a  $3 \times 3$  matrix. Once you know how to calculate the determinant of a  $3 \times 3$  matrix you can calculate the determinant of a  $4 \times 4$  and so on.

A logical question about determinants is where does the procedure come from? Why are determinants defined in the way that they are?

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Click image to the left or use the URL below.

URL: <http://www.ck12.org/flx/render/embeddedobject/61650>

<http://www.youtube.com/watch?v=OI07C1HsOuc> James Sousa: Determinants

### Guidance

The determinant of a matrix  $A$  is written as  $|A|$ . For a  $2 \times 2$  matrix  $A$ , the value is calculated as:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Notice how the diagonals are multiplied and then subtracted.

The determinant of a  $3 \times 3$  matrix is more involved.

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Usually you will start by looking at the top row, although any row or column will work. Then use the checkerboard pattern for signs (shown below) and create smaller  $2 \times 2$  matrices.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$



The smaller  $2 \times 2$  matrices are the entries that remain when the row and column of the coefficient you are working with are ignored.

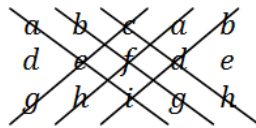
$$\det B = |B| = +a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Next take the determinant of the smaller  $2 \times 2$  matrices and you get a long string of computations.

$$\begin{aligned} &= +a(ei - fh) - b(di - fg) + c(dh - eg) \\ &= aei - afh - bdi + bfg + cdh - ceg \\ &= aei + bfg + cdh - ceg - afh - bdi \end{aligned}$$

Most people do not remember this sequence. A French mathematician named Sarrus demonstrated a great device to memorize the computation of the determinant for  $3 \times 3$  matrices. The first step is simply to copy the first two columns to the right of the matrix. Then draw three diagonal lines going down and to the right.

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$



Notice that they correspond exactly to the three positive terms of the determinant demonstrated above. Next draw three diagonals going up and to the right. These diagonals correspond exactly to the three negative terms.

$$\det B = aei + bfg + cdh - ceg - afh - bdi$$

Sarrus's rule does not work for the determinants of matrices that are not of order  $3 \times 3$ .

### Example A

Find  $\det A$  for  $A = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$

**Solution:**  $\begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix} = 3 \cdot 5 - 2 \cdot 1 = 15 - 2 = 13$

### Example B

Find  $\det B$  for  $B = \begin{bmatrix} 3 & 2 & 1 \\ 5 & 0 & 2 \\ 2 & 1 & 5 \end{bmatrix}$

**Solution:**

$$\begin{aligned} \begin{vmatrix} 3 & 2 & 1 \\ 5 & 0 & 2 \\ 2 & 1 & 5 \end{vmatrix} &= 3 \begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} - 2 \begin{vmatrix} 5 & 2 \\ 2 & 5 \end{vmatrix} + 1 \begin{vmatrix} 5 & 0 \\ 2 & 1 \end{vmatrix} \\ &= 3(0 \cdot 5 - 2 \cdot 1) - 2(5 \cdot 5 - 2 \cdot 2) + 1(5 \cdot 1 - 2 \cdot 0) \\ &= -6 - 42 + 5 = -43 \end{aligned}$$

**Example C**

Find the determinant of  $B$  from example B using Sarrus's Rule.

**Solution:**

$$\begin{array}{ccccc} 3 & 2 & 1 & 3 & 2 \\ 5 & 0 & 2 & 5 & 0 \\ 2 & 1 & 5 & 2 & 1 \end{array}$$

$$\det B = 0 + 8 + 5 - 0 - 6 - 50 = -43$$

As you can see, Sarrus's Rule is efficient and much of the calculations can be done mentally. Additionally, zero values make much of the multiplication easier.

**Concept Problem Revisited**

Determinants for  $2 \times 2$  matrices are defined the way they are because of the general solution to a system of 2 variables and 2 equations.

$$ax + by = e$$

$$cx + dy = f$$

To eliminate the  $x$ , scale the first equation by  $c$  and the second equation by  $a$ .

$$acx + bcy = ec$$

$$acx + ady = af$$

Subtract the second equation from the first and solve for  $y$ .

$$ady - bcy = af - ec$$

$$y(ad - bc) = af - ec$$

$$y = \frac{af - ec}{ad - bc}$$

When you solve for  $x$  you also get  $ad - bc$  in the denominator of the general solution. This pattern led people to start using this strategy in solving systems of equations. The determinant is defined in this way so it will always be the denominator of the general solution of either variable.

**Vocabulary**

The **determinant** of a matrix is a number calculated from the entries in a matrix. The procedure is derived from solving linear systems.

**Sarrus's rule** is a memorization technique that enables you to compute the determinant of  $3 \times 3$  matrices efficiently.

**Guided Practice**

1. Find the determinant of the following matrix.

$$C = \begin{bmatrix} -4 & 12 \\ 1 & -3 \end{bmatrix}$$

2. Find the determinant of the following matrix.

$$D = \begin{bmatrix} 4 & 8 & 3 \\ 0 & 1 & 7 \\ 12 & 5 & 13 \end{bmatrix}$$

3. Find the determinant of the following  $4 \times 4$  matrix by carefully choosing the row or column to work with.

$$E = \begin{bmatrix} 4 & 5 & 0 & 2 \\ -1 & -3 & 0 & 3 \\ 4 & 8 & 1 & 5 \\ -3 & 2 & 0 & 9 \end{bmatrix}$$

**Answers:**

$$1. \det C = \begin{vmatrix} -4 & 12 \\ 1 & -3 \end{vmatrix} = 12 - 12 = 0$$

$$2. \det D = \begin{vmatrix} 4 & 8 & 3 \\ 0 & 1 & 7 \\ 12 & 5 & 13 \end{vmatrix} = 4 \cdot 13 + 8 \cdot 7 \cdot 12 + 0 - 36 - 5 \cdot 7 \cdot 4 - 0 = 548$$

3. Notice that the third column is made up with zeros and a one. Choose this column to make up the coefficients because then instead of having to evaluate the determinant of four individual  $3 \times 3$  matrices, you only need to do one.

$$\begin{aligned} \begin{vmatrix} 4 & 5 & 0 & 2 \\ -1 & -3 & 0 & 3 \\ 4 & 8 & 1 & 5 \\ -3 & 2 & 0 & 9 \end{vmatrix} &= 0 \cdot \begin{vmatrix} -1 & -3 & 3 \\ 4 & 8 & 5 \\ -3 & 2 & 9 \end{vmatrix} - 0 \cdot \begin{vmatrix} 4 & 5 & 2 \\ 4 & 8 & 5 \\ -3 & 2 & 9 \end{vmatrix} + 1 \cdot \begin{vmatrix} 4 & 5 & 2 \\ -1 & -3 & 3 \\ -3 & 2 & 9 \end{vmatrix} - 0 \cdot \begin{vmatrix} 4 & 5 & 2 \\ -1 & -3 & 3 \\ 4 & 8 & 5 \end{vmatrix} \\ &= \begin{vmatrix} 4 & 5 & 2 \\ -1 & -3 & 3 \\ -3 & 2 & 9 \end{vmatrix} \\ &= 4 \cdot (-3) \cdot 9 + 5 \cdot 3 \cdot (-3) + 2 \cdot (-1) \cdot 2 - 18 - 24 - (-45) \\ &= -154 \end{aligned}$$

## Practice

Find the determinants of each of the following matrices.

$$1. \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$$

$$2. \begin{bmatrix} -3 & 6 \\ 2 & 5 \end{bmatrix}$$

$$3. \begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix}$$

$$4. \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$5. \begin{bmatrix} 6 & 5 \\ 2 & -2 \end{bmatrix}$$

$$6. \begin{bmatrix} 1 & 2 \\ 6 & 3 \end{bmatrix}$$

7.  $\begin{bmatrix} -1 & 3 & -4 \\ 4 & 2 & 1 \\ 1 & 2 & 5 \end{bmatrix}$

8.  $\begin{bmatrix} 4 & 5 & 8 \\ 9 & 0 & 1 \\ 0 & 3 & -2 \end{bmatrix}$

9.  $\begin{bmatrix} 0 & 7 & -1 \\ 2 & -3 & 1 \\ 6 & 8 & 0 \end{bmatrix}$

10.  $\begin{bmatrix} 4 & 2 & -3 \\ 2 & 4 & 5 \\ 1 & 8 & 0 \end{bmatrix}$

11.  $\begin{bmatrix} -2 & -6 & -12 \\ -1 & -5 & -2 \\ 2 & 3 & 4 \end{bmatrix}$

12.  $\begin{bmatrix} -2 & 6 & 3 \\ 2 & 4 & 0 \\ -8 & 2 & 1 \end{bmatrix}$

13.  $\begin{bmatrix} 2 & 6 & 4 & 6 \\ 0 & 1 & 0 & 1 \\ 2 & 4 & 2 & 0 \\ -6 & 2 & 3 & 1 \end{bmatrix}$

14.  $\begin{bmatrix} 5 & 0 & 0 & 1 \\ 2 & 1 & 8 & 3 \\ 9 & 3 & 2 & 6 \\ -4 & 2 & 5 & 1 \end{bmatrix}$

15. Can you find the determinant for any matrix? Explain.

16. The following matrix has a determinant of zero:  $\begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix}$ . If the determinant of a matrix is zero, what does that say about the rows of the matrix?

## 7.8 Cramer's Rule

Here you'll use Cramer's Rule to solve systems of linear equations in two and three variables.

At your school book fair, paperbacks cost one price and hardcovers cost another. You buy 3 paperbacks and 2 hardcovers. Your total comes to \$54. Your best friend buys 2 paperbacks and 4 hardcovers. His total comes to \$76. How could you use a matrix to find the price of each type of book?

### Guidance

Previously we learned how to systems of linear equations using graphing, substitution and linear combinations. In this topic, we will explore one way to use matrices and determinants to solve linear systems.

#### Cramer's Rule in Two Variables:

Given the system:

$$\begin{aligned} ax + by &= e \\ cx + dy &= f, \end{aligned}$$

we can set up the matrix  $A$  and solve for  $x$  and  $y$  as shown below:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, x = \frac{\begin{vmatrix} e & d \\ f & b \end{vmatrix}}{\det A}$$

and

$$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\det A},$$

provided  $|A| \neq 0$ .

Note: When  $|A| = 0$ , there is no unique solution. We have to investigate further to determine if there are infinite solutions or no solutions to the system. Notice that there is a pattern here. The coefficients of the variable we are trying to find are replaced with the constants.

#### Cramer's Rule in Three Variables:

Given the system:

$$\begin{aligned} ax + by + cz &= j \\ dx + ey + fz &= k \\ gx + hy + iz &= l, \end{aligned}$$

we can set up the matrix  $A$  and solve for  $x$  and  $y$  as shown below:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} j & b & c \\ k & e & f \\ l & h & i \end{vmatrix}}{\det A}$$

,

$$y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & l & i \end{vmatrix}}{\det A}$$

and

$$z = \frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & l \end{vmatrix}}{\det A}$$

provided  $|A| \neq 0$ . Once again, if the  $|A| = 0$ , then there is no unique solution to the system. Once again there is a pattern here. The coefficients of the variable for which we are trying to solve are replaced with the constants.

### Example A

Use Cramer's Rule to solve the system:

$$\begin{aligned} 3x - 7y &= 13 \\ -5x + 9y &= -19 \end{aligned}$$

#### Solution:

Matrix  $A$  is the matrix made up of the coefficients of  $x$  and  $y$ :

$$A = \begin{bmatrix} 3 & -7 \\ -5 & 9 \end{bmatrix}$$

Now we can find the  $\det A = (3 \cdot 9) - (-7 \cdot -5) = 27 - 35 = -8$

So, using the formulas above:

$$x = \frac{\begin{vmatrix} 13 & -7 \\ -19 & 9 \end{vmatrix}}{-8} = \frac{(13 \cdot 9) - (-7 \cdot -19)}{-8} = \frac{-16}{-8} = 2$$

$$y = \frac{\begin{vmatrix} 3 & 13 \\ -5 & -19 \end{vmatrix}}{-8} = \frac{(3 \cdot -19) - (13 \cdot -5)}{-8} = \frac{-57 - (-65)}{-8} = \frac{8}{-8} = -1$$

Therefore, the solution is (2, -1).

**Example B**

Use Cramer's Rule to solve the system:

$$6x + 3y = -12$$

$$2x + y = 20$$

**Solution:**

Matrix  $A$  is the matrix made up of the coefficients of  $x$  and  $y$ :

$$A = \begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix}$$

Now we can find the  $\det A = (6 \cdot 1) - (3 \cdot 2) = 6 - 6 = 0$

Because the determinant is zero, there is no unique solution and we cannot solve the system further using Cramer's Rule. Looking at the system, we see that the left-hand side of the first equation is a multiple of the second equation, by 3. The right sides are not multiples of each other, therefore there is no solution.

**Example C**

Use Cramer's Rule to solve the system:

$$2x + 2y - z = -7$$

$$5x + y - 2z = -3$$

$$x - 3y + 2z = 21$$

**Solution:**

Matrix  $A$  is the matrix made up of the coefficients of  $x, y$  and  $z$ :

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 5 & 1 & -2 \\ 1 & -3 & 2 \end{bmatrix}$$

Now we can find the determinant of matrix  $A$ :

$$\begin{aligned} \det A &= \begin{vmatrix} 2 & 2 & -1 \\ 5 & 1 & -2 \\ 1 & -3 & 2 \end{vmatrix} \\ &= [(2)(1)(2) + (2)(-2)(1) + (-1)(5)(-3)] - [(1)(1)(-1) + (-3)(-2)(2) + (2)(5)(2)] \\ &= [4 - 4 + 15] - [-1 + 12 + 20] \\ &= 15 - 31 \\ &= -16 \end{aligned}$$

Now using the formulas above, we can find  $x, y$  and  $z$  as follows:

$$x = \frac{\begin{vmatrix} -7 & 2 & -1 \\ -3 & 1 & -2 \\ 21 & -3 & 2 \end{vmatrix}}{-16} = \frac{-32}{-16} = 2 \quad y = \frac{\begin{vmatrix} 2 & -7 & -1 \\ 5 & -3 & -2 \\ 1 & 21 & 2 \end{vmatrix}}{-16} = \frac{48}{-16} = -3 \quad z = \frac{\begin{vmatrix} 2 & 2 & -7 \\ 5 & 1 & -3 \\ 1 & -3 & 21 \end{vmatrix}}{-16} = \frac{-80}{-16} = 5$$

So the solution is (2, -3, 5).

**Intro Problem Revisit** The system of linear equation represented by this situation is:

$$3x + 2y = 54$$

$$2x + 4y = 76$$

We can now set up a matrix and apply Cramer's rule to find the price of each type of book.

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}$$

Now we can find the  $\det A = (3 \cdot 4) - (2 \cdot 2) = 12 - 4 = 8$

So, using the formulas above:

$$x = \frac{\begin{vmatrix} 54 & 2 \\ 76 & 4 \end{vmatrix}}{8} = \frac{(54 \cdot 4) - (76 \cdot 2)}{8} = \frac{216 - 152}{8} = 8$$

$$y = \frac{\begin{vmatrix} 3 & 54 \\ 2 & 76 \end{vmatrix}}{8} = \frac{(3 \cdot 76) - (2 \cdot 54)}{8} = \frac{(228 - 108)}{8} = 15$$

Therefore, paperbacks cost \$8 and hardcovers cost \$15.

### Guided Practice

Use Cramer's Rule to solve the systems below.

1.

$$2x + 5y = 7$$

$$x + 3y = 2$$

2.

$$4x - y = 6$$

$$-8x + 2y = 10$$

3.

$$x + 2y + 3z = 8$$

$$2x - y + 4z = 3$$

$$-x - 4y + 3z = 14$$



**Answers**

1. Find the

$$\det A = \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} = 1.$$

Now find  $x$  and  $y$  as follows:

$$x = \frac{\begin{vmatrix} 7 & 5 \\ 2 & 3 \end{vmatrix}}{1} = \frac{11}{1} = 11 \quad y = \frac{\begin{vmatrix} 2 & 7 \\ 1 & 2 \end{vmatrix}}{1} = \frac{-3}{1} = -3, \quad \text{solution: } (11, -3)$$

2. Find the

$$\det A = \begin{vmatrix} 4 & -1 \\ -8 & 2 \end{vmatrix} = 8 - 8 = 0.$$

Therefore, there is no unique solution. We must use linear combination or the substitution method to determine whether there are an infinite number of solutions or no solutions. Using linear combinations we can multiply the first equation by 2 and get the following:

$$\begin{array}{rcl} 8x - 2y = 12 \\ -8x + 2y = 10 \\ \hline 0 = 22 \end{array} \Rightarrow \text{Therefore, there is no solution.}$$

3. Find the

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ -1 & -4 & 3 \end{vmatrix} = -34.$$

Now find  $x, y$  and  $z$  as follows:

$$x = \frac{\begin{vmatrix} 8 & 2 & 3 \\ 3 & -1 & 4 \\ 14 & -4 & 3 \end{vmatrix}}{-34} = \frac{204}{-34} = -6 \quad y = \frac{\begin{vmatrix} 1 & 8 & 3 \\ 2 & 3 & 4 \\ -1 & 14 & 3 \end{vmatrix}}{-34} = \frac{-34}{-34} = 1 \quad z = \frac{\begin{vmatrix} 1 & 2 & 8 \\ 2 & -1 & 3 \\ -1 & -4 & 14 \end{vmatrix}}{-34} = \frac{-136}{-34} = 4$$

Therefore, the solution is  $(-6, 1, 4)$ .**Explore More**

Solve the systems below using Cramer's Rule. If there is no unique solution, use an alternate method to determine whether the system has infinite solutions or no solution.

1.

$$\begin{aligned}5x - y &= 22 \\ -x + 6y &= -16\end{aligned}$$

2.

$$\begin{aligned}2x + 5y &= -1 \\ -3x - 8y &= 1\end{aligned}$$

3.

$$\begin{aligned}4x - 3y &= 0 \\ -6x + 9y &= 3\end{aligned}$$

4.

$$\begin{aligned}4x - 9y &= -20 \\ -5x + 15y &= 25\end{aligned}$$

5.

$$\begin{aligned}2x - 3y &= -4 \\ 8x + 12y &= -24\end{aligned}$$

6.

$$\begin{aligned}3x - 7y &= 12 \\ -6x + 14y &= -24\end{aligned}$$

7.

$$\begin{aligned}x + 5y &= 8 \\ -2x - 10y &= 16\end{aligned}$$

8.

$$\begin{aligned}2x - y &= -5 \\ -3x + 2y &= 13\end{aligned}$$

9.

$$\begin{aligned}x + y &= -1 \\ -3x - 2y &= 6\end{aligned}$$

Solve the systems below using Cramer's Rule. You may wish to use your calculator to evaluate the determinants.

10.

$$\begin{aligned}3x - y + 2z &= 11 \\ -2x + 4y + 13z &= -3 \\ x + 2y + 9z &= 5\end{aligned}$$

11.

$$\begin{aligned}3x + 2y - 7z &= 1 \\ -4x - 3y + 11z &= -2 \\ x + 4y - z &= 7\end{aligned}$$

12.

$$\begin{aligned}6x - 9y + z &= -6 \\ 4x + 3y - 2z &= 10 \\ -2x + 6y + z &= 0\end{aligned}$$

13.

$$\begin{aligned}x - 2y + 3z &= 5 \\ 4x - y + 4z &= 14 \\ 5x + 2y - 4z &= -3\end{aligned}$$

14.

$$\begin{aligned}-3x + y + z &= 10 \\ 2x + y - 2z &= 15 \\ -4x - 2y + 4z &= -20\end{aligned}$$

15. The Smith and Jamison families go to the county fair. The Smiths purchase 6 corndogs and 3 cotton candies for \$21.75. The Jamisons purchase 3 corndogs and 4 cotton candies for \$15.25. Write a system of linear equations and solve it using Cramer's Rule to find the price of each food.

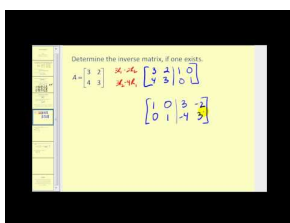
## 7.9 Inverse Matrices

Here you will learn how to find the inverse of a matrix and how to solve a system of equations using an inverse matrix.

Two numbers are multiplicative inverses if their product is 1. Every number besides the number 0 has a multiplicative inverse. For matrices, two matrices are inverses of each other if they multiply to be the identity matrix.

What kinds of matrices do not have inverses?

### Watch This



### MEDIA

Click image to the left or use the URL below.

URL: <http://www.ck12.org/flx/render/embeddedobject/61656>

<http://www.youtube.com/watch?v=KBYvP6YG58g> James Sousa: Inverse Matrices Using Augmented Matrices

### Guidance

Consider a matrix  $A$  that has inverse  $A^{-1}$ . How do you find matrix  $A^{-1}$  if you just have matrix  $A$ ?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}, A^{-1} = ?$$

The answer is that you augment matrix  $A$  with the identity matrix and row reduce.

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right]$$

The row reducing is demonstrated in Example A. The right part of the augmented Matrix is the inverse matrix  $A^{-1}$ .

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{1}{6} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{6} & 0 \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

Fractions are usually unavoidable when computing inverses.

One reason why inverses are so powerful is because they allow you to solve systems of equations with the same logic as you would solve a single linear equation. Consider the following system based on the coefficients of matrix  $A$  from above.

$$\begin{aligned}x + 2y + 3z &= 96 \\x + 0y + z &= 36 \\0x + 2y - z &= -12\end{aligned}$$

By writing this system as a matrix equation you get:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$$

$$A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$$

If this were a normal linear equation where you had a constant times the variable equals a constant, you would multiply both sides by the multiplicative inverse of the coefficient. Do the same in this case.

$$A^{-1} \cdot A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$$

All that is left is for you to perform the matrix multiplication to get the solution. See Example B.

### Example A

Show the steps for finding the inverse matrix  $A$  from the guidance section.

**Solution:**

$$\begin{aligned}&\left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{-I} \left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & -2 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{+II} \left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & -2 & -1 & 1 & 0 \\ 0 & 0 & -3 & -1 & 1 & 1 \end{array} \right] \xrightarrow{\div(-2)} \left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -3 & -1 & 1 & 1 \end{array} \right] \xrightarrow{\div(-3)} \left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array} \right] \xrightarrow{-2III} \left[ \begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array} \right] \xrightarrow{-\frac{1}{6}II} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array} \right]\end{aligned}$$

The matrix on the right is the inverse matrix  $A^{-1}$ .

$$A^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

### Example B

Solve the following system of equations using inverse matrices.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$$

**Solution:**

$$\begin{aligned} A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix} \\ A^{-1} \cdot A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= A^{-1} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix} \\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= A^{-1} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix} \\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix} \\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} -\frac{1}{3} \cdot 96 + \frac{4}{3} \cdot 36 + \frac{1}{3} \cdot (-12) \\ \frac{1}{6} \cdot 96 - \frac{1}{6} \cdot 36 + \frac{1}{3} \cdot (-12) \\ \frac{1}{3} \cdot 96 - \frac{1}{3} \cdot 36 - \frac{1}{3} \cdot (-12) \end{bmatrix} \\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 12 \\ 6 \\ 24 \end{bmatrix} \end{aligned}$$

### Example C

Find the inverse of the following matrix.

$$\begin{bmatrix} 1 & 6 \\ 4 & 24 \end{bmatrix}$$

**Solution:**

$$\left[ \begin{array}{cc|cc} 1 & 6 & 1 & 0 \\ 4 & 24 & 0 & 1 \end{array} \right] \rightarrow \begin{array}{c} \rightarrow \\ -4I \end{array} \rightarrow \left[ \begin{array}{cc|cc} 1 & 6 & 1 & 0 \\ 0 & 0 & -4 & 1 \end{array} \right]$$

This matrix is not invertible because its rows are not linearly independent. To test to see if a square matrix is invertible, check whether or not the determinant is zero. If the determinant is zero then the matrix is not invertible because the rows are not linearly independent.

### Concept Problem Revisited

Non-square matrices do not generally have inverses. Square matrices that have determinants equal to zero do not have inverses.

### Vocabulary

**Multiplicative inverses** are two numbers or matrices whose product is one or the identity matrix.

### Guided Practice

1. Confirm matrix  $A$  and  $A^{-1}$  are inverses by computing  $A^{-1} \cdot A$  and  $A \cdot A^{-1}$ .

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}, A^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

2. Use a calculator to compute  $A^{-1}$ , compute  $A^{-1} \cdot A$ , compute  $A \cdot A^{-1}$  and compute  $A^{-1} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$ .
3. The identity matrix happens to be its own inverse. Find another matrix that is its own inverse.

**Answers:**

1.

$$A^{-1} \cdot A = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} =$$

$$a_{11} = -\frac{1}{3} \cdot 1 + \frac{4}{3} \cdot 1 + \frac{1}{3} \cdot 0 = 1$$

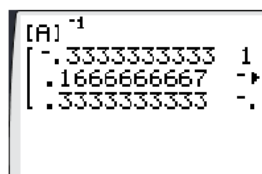
$$a_{22} = \frac{1}{6} \cdot 2 - \frac{1}{6} \cdot 0 + \frac{1}{3} \cdot 2 = 1$$

$$a_{33} = \frac{1}{3} \cdot 3 - \frac{1}{3} \cdot 1 - \frac{1}{3}(-1) = 1$$

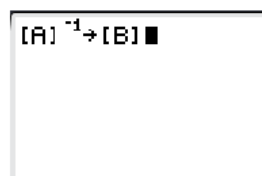
Note that the rest of the entries turn out to be zero. This is left for you to confirm.

2. Start by entering just matrix  $A$  into the calculator.

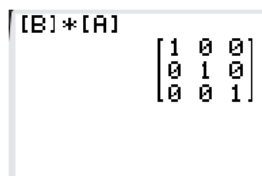
To compute matrix  $A^{-1}$  use the inverse button programmed into the calculator. Do not try to raise the matrix to the negative one exponent. This will not work.



Note that the calculator may return decimal versions of the fractions and will not show the entire matrix on its limited display. You will have to scroll to the right to confirm that  $A^{-1}$  matches what you have already found. Once you have found  $A^{-1}$  go ahead and store it as matrix  $B$  so you do not need to type in the entries.



$$A^{-1} \cdot A = B \cdot A$$



$$A \cdot A^{-1} = A \cdot B$$

A calculator screen showing the operation  $[A] \cdot [B]$ . The result is a 3x3 identity matrix:  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

$$A^{-1} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix} = B \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix} = B \cdot C$$

You need to create matrix  $C = \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$

A calculator screen showing the operation  $[B] \cdot [C]$ . The result is a 3x1 column vector:  $\begin{bmatrix} 12 \\ 6 \\ 24 \end{bmatrix}$ .

Being able to effectively use a calculator should improve your understanding of matrices and allow you to check all the work you do by hand.

3. Helmert came up with a very clever matrix that happens to be its own inverse. Here are the  $2 \times 2$  and the  $3 \times 3$  versions.

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

### Practice

Find the inverse of each of the following matrices, if possible. Make sure to do some by hand and some with your calculator.

1.  $\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$

2.  $\begin{bmatrix} -3 & 6 \\ 2 & 5 \end{bmatrix}$

3.  $\begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix}$

4.  $\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$

5.  $\begin{bmatrix} 6 & 5 \\ 2 & -2 \end{bmatrix}$

6.  $\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$



$$7. \begin{bmatrix} -1 & 3 & -4 \\ 4 & 2 & 1 \\ 1 & 2 & 5 \end{bmatrix}$$

$$8. \begin{bmatrix} 4 & 5 & 8 \\ 9 & 0 & 1 \\ 0 & 3 & -2 \end{bmatrix}$$

$$9. \begin{bmatrix} 0 & 7 & -1 \\ 2 & -3 & 1 \\ 6 & 8 & 0 \end{bmatrix}$$

$$10. \begin{bmatrix} 4 & 2 & -3 \\ 2 & 4 & 5 \\ 1 & 8 & 0 \end{bmatrix}$$

$$11. \begin{bmatrix} -2 & -6 & -12 \\ -1 & -5 & -2 \\ 2 & 3 & 4 \end{bmatrix}$$

$$12. \begin{bmatrix} -2 & 6 & 3 \\ 2 & 4 & 0 \\ -8 & 2 & 1 \end{bmatrix}$$

$$13. \text{ Show that Helmert's } 2 \times 2 \text{ matrix is its own inverse: } \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$14. \text{ Show that Helmert's } 3 \times 3 \text{ matrix is its own inverse: } \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}.$$

15. Non-square matrices sometimes have left inverses, where  $A^{-1} \cdot A = I$ , or right inverses, where  $A \cdot A^{-1} = I$ . Why can't non-square matrices have "regular" inverses?

A system of equations is a collection of two or more equations with multiple unknown variables. You learned that solving a system means solving for these variables. Graphically this means finding where the lines, curves or planes intersect. You learned that a matrix is a rectangular array of numbers that corresponds to the coefficients in a system. Using a matrix to represent the system is incredibly powerful because it makes it easier to solve for the variables. In this chapter on systems and matrices, you reviewed techniques for solving systems in two and three dimensions, learned the basic idea of matrices, exercised different properties of matrices and finally used matrices to solve systems.

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## 7.10 References

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