# CK-12 FlexBook

# Trigonometry

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Printed: July 10, 2015

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## Chapter Outline

# 1.1 Introduction to Probability

## Introduction

**What is probability?**

***Probability* is the likelihood that an event will occur.** It is a mathematical way of calculating how likely an **event** is likely to occur. **An *event* is a result of an experiment or activity** that might include such things as:

- flipping a coin
- spinning a spinner
- rolling a number cube
- choosing an item from a jar or bag

**How do we calculate probability?**

We calculate probability by looking at the ratio of ***favorable outcomes*** to ***total outcomes*** in a given situation. In ratio form, the probability of an event is:

$P(\text{event}) = \text{favorable outcomes} : \text{total outcomes}$

***Write this ratio down in your notebook.***

$$\textit{Probability} = \frac{\textbf{The number of favorable outcomes}}{\textbf{The total number of possible outcomes}}$$

**NOTES:**

**1) The probability of a certain event is 1.**

**2) The probability of an impossible event is 0.**

**3) The probability of an event is always a fraction or decimal between 0 and 1.**

View this video for an introduction to probability:

**An introduction to probability** http://www.educreations.com/lesson/view/probability/9491853/?ref=app

An ***outcome*** is a possible result of some event occurring. For example, when you flip a coin, "heads" is one outcome; tails is a second outcome. ***Total outcomes*** are computed simply by counting all possible outcomes.

**Chapter Note:** Keep in mind as you go through this chapter that all outcomes used are presumed to be "fair." That is – when you flip a coin, the outcomes of heads or tails are equally likely. When you spin a spinner, sections are all of equal size and equally likely to be landed on. When you toss a number cube, faces of the cube are the same size and again equally likely to be landed on. And so on.

**For flipping a coin:**

$$\begin{aligned}\text{total outcomes} &= \text{heads, tails}\\ &= 2 \text{ total outcomes}\end{aligned}$$

**For tossing a number cube:**

$$\begin{aligned}\text{total outcomes} &= \cdot\, 1 \,\cdot\cdot\, 2 \,\cdots\, 3 \,\cdot\cdot\cdot\cdot\, 4 \,\cdot\cdot\cdot\cdot\cdot\, 5 \,\cdot\cdot\cdot\cdot\cdot\cdot\, 6\\ &= 6 \text{ total outcomes}\end{aligned}$$

**For selecting a day of the week:**

$$\begin{aligned}\text{total outcomes} &= \text{Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}\\ &= 7 \text{ total outcomes}\end{aligned}$$

***Favorable outcomes* are the specific outcomes you are looking for.**

**For flipping a coin and having it come up heads:**

$$\begin{aligned}\text{favorable outcomes} &= \text{heads}\\ &= 1 \text{ favorable outcome}\end{aligned}$$

**For tossing a number cube and having it come with up an even number:**

$$\begin{aligned}\text{favorable outcomes} &= \cdot\, 1 \,\cdot\cdot\, 2 \,\cdots\, 3 \,\cdot\cdot\cdot\cdot\, 4 \,\cdot\cdot\cdot\cdot\cdot\, 5 \,\cdot\cdot\cdot\cdot\cdot\cdot\, 6\\ &= 3 \text{ favorable outcomes}\end{aligned}$$

**For randomly choosing a date and have it land on a weekday:**

$$\begin{aligned}\text{favorable outcomes} &= \text{Monday, Tuesday, Wednesday, Thursday, Friday}\\ &= 5 \text{ favorable outcomes}\end{aligned}$$

**To write a ratio, we compare the favorable outcome to the total outcomes. Comparing favorable outcomes to possible total outcomes is what we call *theoretical probability.***

I

Look at this example.

Example

What is the probability of flipping heads on a coin?

**To work through this probability, we are going to be writing a ratio that compares the number of favorable outcomes with the total number of outcomes.**

**Favorable outcome = 1, since there is one heads on a coin**

**Total outcomes = 2, since there is the possibility of heads or tails**

**The answer is 1/2.**

Example

**For tossing a number cube and having it land an even number:**

P (even) = favorable outcomes : total outcomes

= 3 favorable outcomes : 6 total outcomes

= 3 : 6

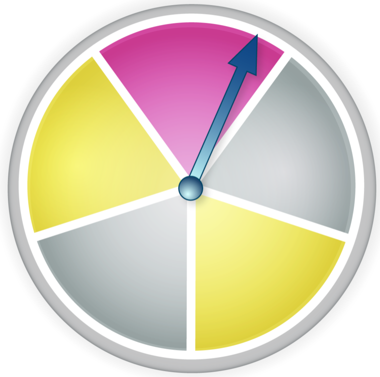
= 1 : 2

Note that 3:6 can be simplifed to 1:2.

**Our final answer is 1/2.**

**To find any probability, follow the steps below.**

Problem: What is the probability of the arrow landing on a yellow section?



**Step 1: Count the number of favorable outcomes.**

There are 2 yellow spaces, so

favorable outcomes = 2

**Step 2: Count the number of total outcomes.**

There are 5 spaces in all, so

total outcomes = 5

**Step 3: Write the ratio of favorable outcomes to total outcomes**

$$\begin{aligned} P(\text{yellow}) &= \text{favorable outcomes} : \text{total outcomes} \\ &= 2:5 \end{aligned}$$

**The answer is 2/5**

***Take a few minutes to write these steps in your notebook.***

**Now let's apply these steps to an example.**

Example

What is the probability of the arrow landing on a silver or pink section?

**Step 1: Count the number of favorable outcomes.**

There are 2 silver spaces and 1 pink space, so

favorable outcomes = 3

**Step 2: Count the number of total outcomes.**

There are 5 spaces in all, so

total outcomes = 5

**Step 3: Write the ratio of favorable outcomes to total outcomes**

$P(\text{silver or pink}) = \text{favorable outcomes : total outcomes} = 3:5$

**Our answer is 3/5.**

Example

A bag contains 5 black ping pong balls, 8 white ping pong balls, and 7 yellow ping pong balls. What is the probability of drawing a black ping pong ball from the bag?

**First, let's look at writing a ratio.**

$$\begin{aligned}
P(\text{black}) &= \frac{\text{favorable outcomes}}{\text{total outcomes}}\\
\text{favorable outcomes} &= 5\\
\text{total outcomes} &= 20\\
P(\text{black}) &= 5:20\\
&= 1:4
\end{aligned}$$

**Now that we have a ratio, we can easily take this ratio and write it as a fraction. Notice that the first value in the ratio becomes the numerator and the second number becomes the denominator.**

**1:4 becomes** $\frac{1}{4}$

**Next, we can take this and convert it to a decimal. There are two ways to do this.**

**The first way is to divide the numerator by the denominator.**

$$4\overline{)1.00}^{\,.25}$$

**The decimal is .25.**

**The proportion shows us how to convert the decimal to a percent easily.**

**.25 or** $\frac{25}{100} = 25\%$

**Now we have written the probability all possible ways.**

Let's look at another example.

Example

Thinking about the bag with the ping pong balls,

A bag contains 5 black ping pong balls, 8 white ping pong balls, and 7 yellow ping pong balls.

what is the probability of choosing a yellow ping pong ball?

$$\begin{aligned} P(\text{yellow}) &= \frac{\text{favorable outcomes}}{\text{total outcomes}} \\ \text{favorable outcomes} &= 7 \\ \text{total outcomes} &= 20 \\ P(\text{yellow}) &= 7:20 \end{aligned}$$

**Now let's write this as a fraction.**

$7:20 = \frac{7}{20}$

**As a decimal:**

$\frac{7}{20} = \frac{35}{100} = .35$

**And as a percent:**

$.35 = 35\%$

***Remember you can also convert a decimal to a percent by moving the decimal point two places to the right and then adding a percent sign.***

**1**

Vocabulary

Here are the vocabulary words that are found in this lesson.

**Probability**
the likelihood that an event will happen.

**Event**
result of an experiment or an activity

**Favorable Outcome**
the outcome that you are looking for

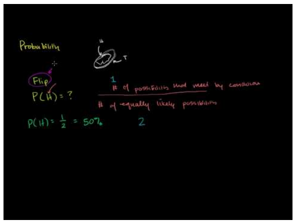
**Total Outcomes**
the total number of possible outcomes

**Ratio**
a comparison of two quantities

**Theoretical Probability**
the ratio that compares the number of favorable outcomes to the number of total outcomes.

**Prediction**
a reasonable guess about a future event.

## Technology Integration



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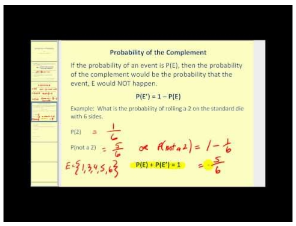
Khan Academy, Basic Probability

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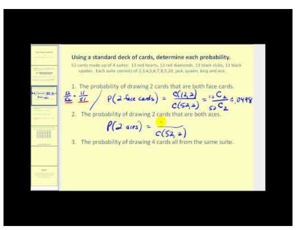
James Sousa, Introduction to Probability

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James Sousa, Determining Probability

## Time to Practice

Directions: Answer each question or solve each problem as it connects to probability.

1. For rolling a 4 on the number cube:

(a) List each favorable outcome.

(b) Count the number of favorable outcomes.

(c) Write the total number of outcomes.

2. For rolling a number greater than 2 on the number cube:

(a) List each favorable outcome.

(b) Count the number of favorable outcomes.

(c) Write the total number of outcomes.

3. For rolling a 5 or 6 on a number cube:

(a) List each favorable outcome.

(b) Count the number of favorable outcomes.

(c) Write the total number of outcomes.

4. A box contains 12 slips of paper numbered 1 to 12. For randomly choosing a slip with an even number on it:

(a) List each favorable outcome.

(b) Count the number of favorable outcomes.

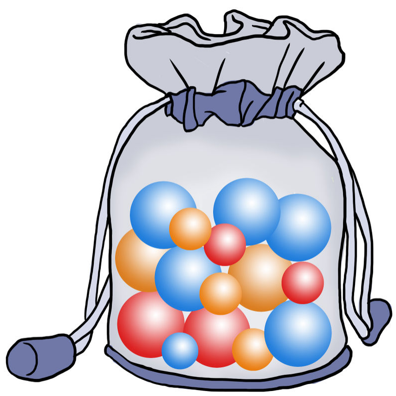
(c) Write the total number of outcomes.

5. A box contains 12 slips of paper numbered 1 to 12. For randomly choosing a slip with a number greater than 3:

(a) List each favorable outcome.

(b) Count the number of favorable outcomes.

(c) Write the total number of outcomes.



6. For randomly choosing a marble and having it turn out to be orange:

(a) Count the number of favorable outcomes.

(b) Write the total number of outcomes.

7. For randomly choosing a marble and having it turn out to be large:

(a) Count the number of favorable outcomes.

(b) Write the total number of outcomes.

8. For randomly choosing a marble and having it turn out to be blue:

(a) Count the number of favorable outcomes.

(b) Write the total number of outcomes.

9. For randomly choosing a marble and having it turn out to be small:

(a) Count the number of favorable outcomes.

(b) Write the total number of outcomes.

10. For randomly choosing a marble and having it turn out to be orange and large:

(a) Count the number of favorable outcomes.

(b) Write the total number of outcomes.


11. What is the probability of the spinner landing on 9?

(a) List each favorable outcome.

(b) Count the number of favorable outcomes.

(c) Count the total outcomes.

(d) Write the probability. Simplify, if necessary.

12. What is the probability of the spinner landing on 3 or 4?

(a) List each favorable outcome.

(b) Count the number of favorable outcomes.

(c) Count the total outcomes.

(d) Write the probability. Simplify, if necessary.

13. What is the probability of the spinner landing on blue?

(a) List each favorable outcome.

(b) Count the number of favorable outcomes.

(c) Count the total outcomes.

(d) Write the probability. Simplify, if necessary.

14. What is the probability of the spinner landing on a silver number greater than 4?

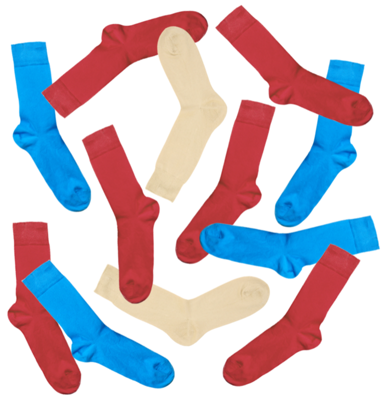
(a) List each favorable outcome.

(b) Count the number of favorable outcomes.

(c) Count the total outcomes.

(d) Write the probability. Simplify, if necessary.

Directions: Answer each question and write the probability as a fraction, a decimal and a percent.



15. A clothes dryer contains 12 socks. What is the probability of reaching inside the dryer and pulling out a blue sock?

(a) List each favorable outcome.

(b) Count the number of favorable outcomes.

(c) Write the probability.

16. What is the probability of pulling a red sock out of the dryer?

(a) List each favorable outcome.

(b) Count the number of favorable outcomes.

(c) Write the probability.

17. What is the probability of pulling a blue or white sock out of the dryer?

(a) List each favorable outcome.

(b) Count the number of favorable outcomes.

(c) Write the probability.

18. What is the probability of pulling a blue or red sock out of the dryer?

(a) List each favorable outcome.

(b) Count the number of favorable outcomes.

(c) Write the probability.

# 1.2 Probability with a deck of cards

## Introduction

**Many probability questions deal with selecting one or more playing cards. Here are some facts you need to know about a deck of cards:**

**1) There are 52 cards in a deck.**

**2) There are 4 suits.**

**The two red suits are hearts and diamonds.**

**The two black suits are spades and clubs.**

**3) Each suit contains 13 cards:**

**ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, king**

**View this video for a detailed explanation of a deck of cards:**

**Probability with a deck of cards** http://www.educreations.com/lesson/view/probability-with-cards/9492184/?ref=app

**A card is selected from a standard deck of cards. Find the following probabilities:**

**a) P(nine)**

**b) P(red queen)**

**c) P(three of spades)**

**a) P(nine) =**

**How many nines are there in a deck ?**

$$\frac{\text{\# of favorable outcomes}}{\text{total \# of outcomes}} = \frac{4}{52} = \frac{1}{13}.$$

**b) P(red queen)**

**How many red queens are there in a deck?**

**Two – queen of hearts**

**queen of diamonds**

$$\frac{\text{\# of favorable outcomes}}{\text{total \# of outcomes}} = \frac{2}{52} = \frac{1}{26}.$$

**c) P(three of spades)**

**How many three of spades are there ?**

**Just one**

$$\frac{\text{\# of favorable outcomes}}{\text{total \# of outcomes}} = \frac{1}{52}.$$

# 1.3 Compound Probability

## Introduction

3 red marbles, 1 green marble, and 4 blue marbles are in a bag and two marbles are selected (the $1^{st}$ marble is not replaced before the second is drawn). Find:

**a) P(selecting 2 red marbles)**

**b) P(selecting 2 blue marbles)**

**c) P(selecting 2 green marbles)**

**a) P(selecting 2 red marbles)**

**On the first pick, what is the P(a red marble)**

$$\frac{3}{8}$$

**Now assume you chose a red marble with pick # 1.**

**What is the probability of getting a second red marble?**

**2 –> Only two red marbles remain**

**7 –> Only seven marbles are left**

**When you are finding more than one "event" in probability (picking 2 marbles) multiply the probabilities**

$$\frac{3}{8} \cdot \frac{2}{7} = \frac{6}{56} = \frac{3}{28}$$

**b) P(selecting 2 blue marbles)**

3 red marbles, 1 green marble, and 4 blue marbles

**Probability of getting a blue on the first pick**

$$\frac{4}{8}$$

**Assume you got a blue on the first pick. Now what is the probability of getting a blue on the second pick?**

$$\frac{3}{7}$$

**Now multiply:**

$$\frac{4}{8} \cdot \frac{3}{7} = \frac{12}{56} = \frac{3}{14}$$

**c) P(2 greens)**

3 red marbles, 1 green marble, and
4 blue marbles

$$\frac{1}{8} \cdot \frac{0}{7} = \frac{0}{56} = 0$$ **<– since there is no chance, the probability is 0.**

**Compound probability is sometimes a challenging topic in math. For more explanation, view the following video:**

**Compound probability** http://www.educreations.com/lesson/view/compound-probability/9569336/?ref=app

**Here is a video which contains more compound probability examples:**

Compound Probability examples http://youtu.be/kfBVOjAvHNc

# 1.4 Finding Outcomes

## Introduction

***The Talent Show Outfit***

Alicia is going to sing for the Talent Show. She is very excited and has selected a wonderful song to sing. She has been practicing with her singing teacher for weeks and is feeling very confident about her ability to do a wonderful job.

Her performance outfit is another matter. Alicia has selected a few different skirts and a few different shirts and shoes to wear. Here are her options for shirts

Striped shirt

Solid shirt

Here are her options for skirts.

Blue skirt

Red skirt

Brown skirt

Here are her options for shoes

Dance shoes

Black dress shoes

How many different outfits can Alicia create given these options?

**This is best done using a tree diagram. Alicia needs to organize her clothing options using a tree diagram. This lesson will show you all about tree diagrams. When finished, you will know how many possible outfits Alicia can create.**

***What You Will Learn***

In this lesson you will learn how to correctly apply the following skills.

- Use tree diagrams to list all possible outcomes of a series of events involving two or more choices or results.
- Recognize all possible outcomes of an experiment as the sample space.
- Find probability of specified outcomes using tree diagrams.

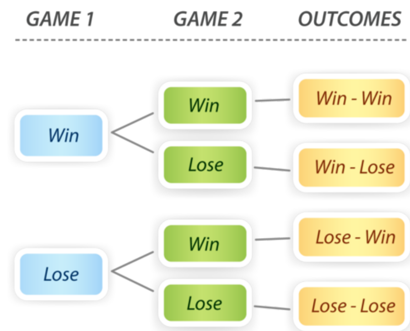
***Teaching Time***

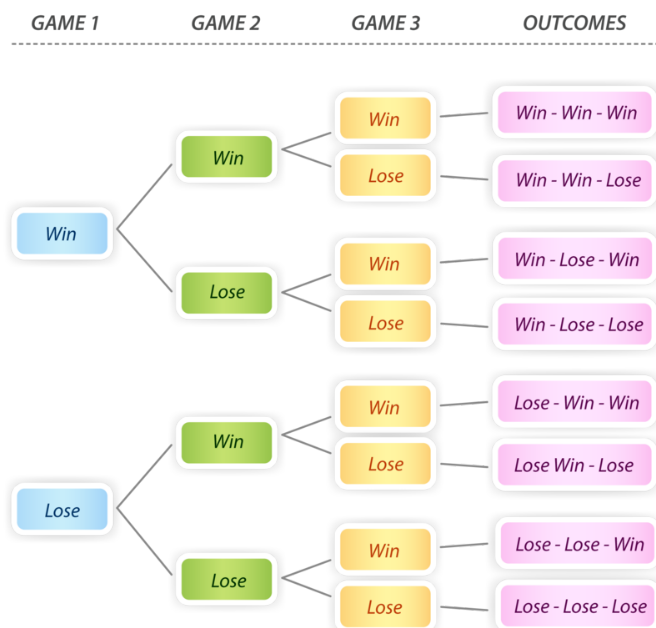
I. **Use Tree Diagrams to List all Possible Outcomes of a Series of Events Involving Two or More Choices or Results**

Nadia's soccer team has 2 games to play this weekend. How many outcomes are there for Nadia's team?

A good way to find the total number of outcomes for events is to make a ***tree diagram.*** **A *tree diagram* is a branching diagram that shows all possible outcomes for an event.**

To make a tree diagram, split the different events into either-or choices. You can list the choices in any order. Here is a tree diagram for game 1 and game 2.




As you can see, there are four different outcomes for the two games:

win-win win-lose
lose-win lose-lose

What happens when you increase the number of games to three? Just add another section to your tree diagram.

**In all, there are 8 total outcomes.**

**A tree diagram is a great way to visually see all of the possible options. It can also help you to organize your ideas so that you don't miss any possibilities.**

Example

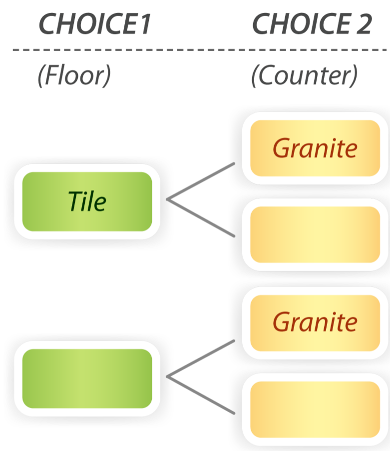
To remodel her kitchen, Gretchen has the following choices: Floor: tile or wood; Counter: Granite or formica; Sink: white, steel, stone. How many different choices can Gretchen make?

**First, let's create a tree diagram that shows all of the possible options.**

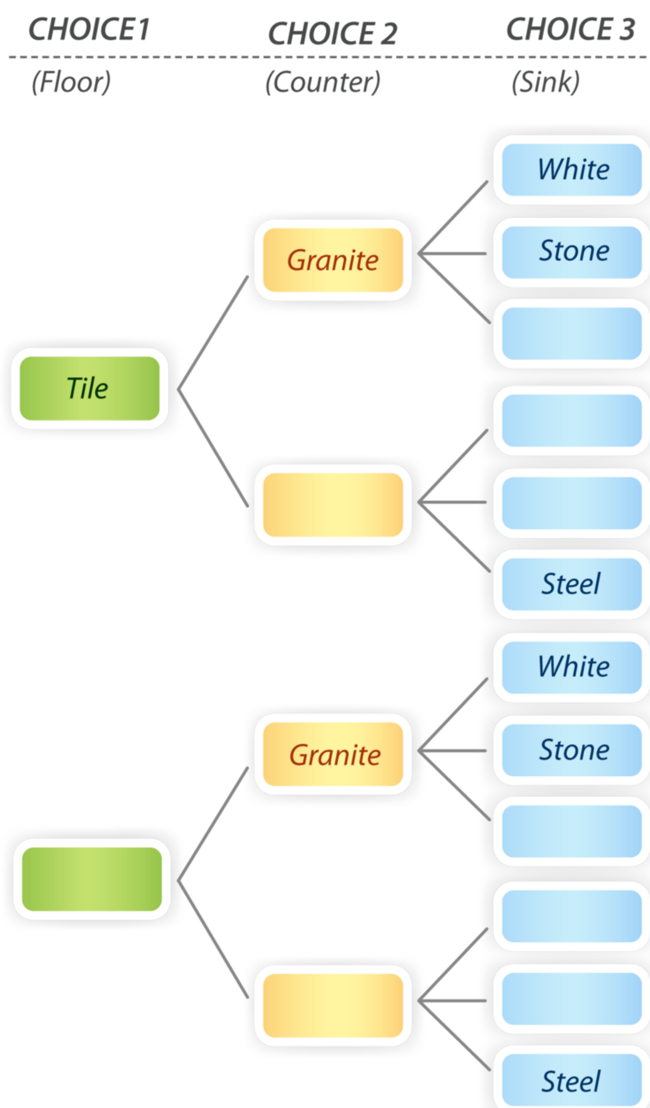
**Step 1:** List the choices.

Choice 1 Floor: tile, wood Choice 2 Counter: granite, formica Choice 3 Sink: white, steel, stone

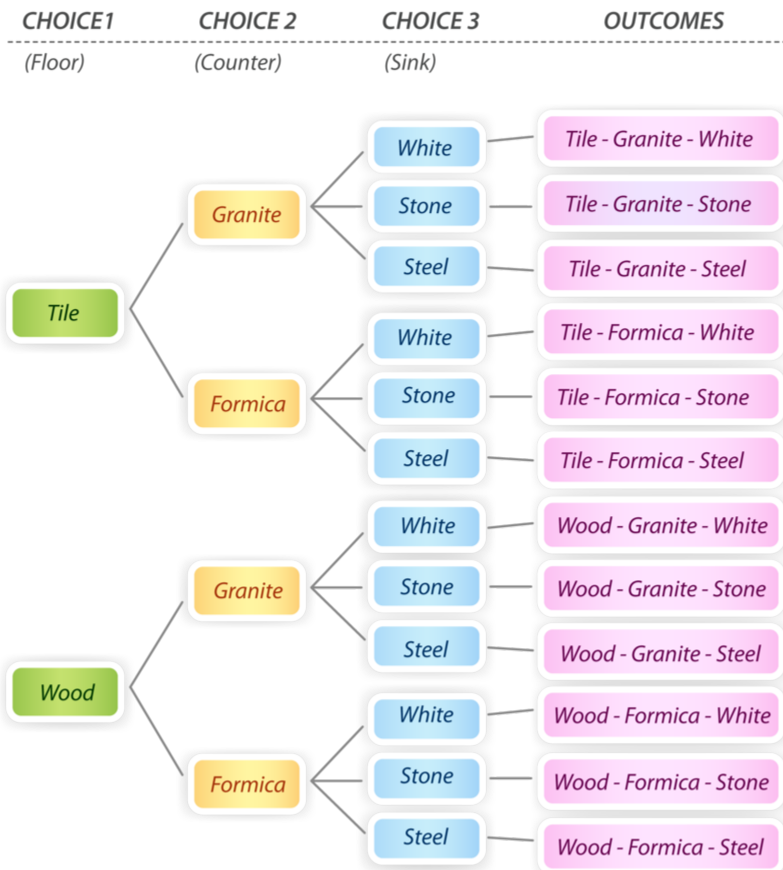
**Step 2:** Start the tree diagram by listing any of the choices for Choice 1. Then have Choice 1 branch off to Choice 2. Make sure Choice 2 repeats for each branch of Choice 1. Can you identify the missing labels in the tree below?




**Step 3:** Fill in the third choice. We have left some of the spaces for you to fill in.




**Step 4:** Fill in the outcomes. Again some of the spaces are left for you to fill in.




**You can see that there are 12 possible outcomes for the kitchen design.**

II. **Recognize all Possible Outcomes of an Experiment as the Sample Space**

When you conduct an experiment, there may be few or many possible outcomes. For example, if you are doing an experiment with a coin, there are two possible outcomes because there are two sides of the coin. You can either have heads or tails. If you have an experiment with a number cube, there are six possible outcomes, because there are six sides of the number cube and the sides are numbered one to six. We can think of all of these possible outcomes as the ***sample space***.

**A *sample space* is the set of all possible outcomes for a probability experiment or activity.** For example, on the spinner there are 5 different colors on which the arrow can land. The sample space, $S$, for one spin of the spinner is then:

$S =$ red, yellow, pink, green, blue

These are the only outcomes that result from a single spin of the spinner.

Changing the spinner changes the sample space. This second spinner still has 5 equal-sized sections. But its sample space now has only 3 outcomes:

$S =$ red, yellow, blue

Let's look at an example having to do with sample spaces.

Example

A small jar contains 1 white, 1 black, and 1 red marble. If one marble is randomly chosen, how many possible outcomes are there in the sample space?

**Since only a single marble is being chosen, the total number of possible outcomes, the sample space, matches the marble colors.**

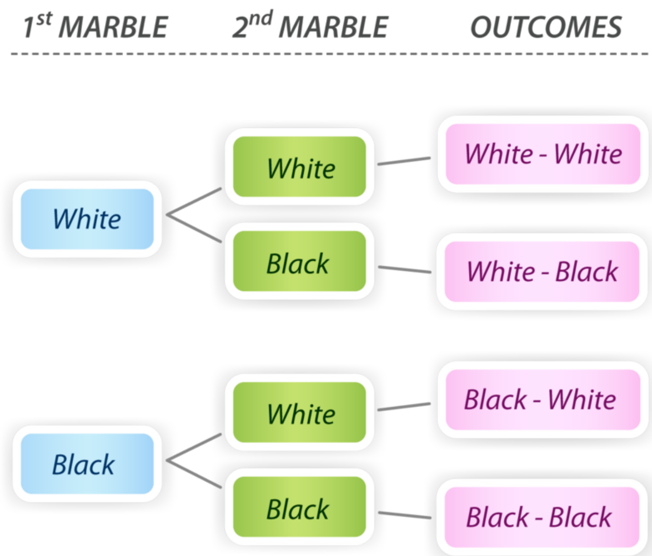
$S =$ white, black, red

**Sometimes, the sample space can change if an experiment is performed more than once. If a marble is selected from a jar and then replaced and if the experiment is conducted again, then the sample space can change. The number of outcomes is altered. When this happens, we can use tree diagrams to help us figure out the number of outcomes in the sample space.**

Example

A jar contains 1 white and 1 black marble. If one marble is randomly chosen, returned to the jar, then a second marble is chosen, how many possible outcomes are there?

This is an example where a tree diagram is very useful. Consider the marbles one at a time. After the first marble is chosen, it is returned to the jar so now there are again two choices for the second marble. Use a tree diagram to list the outcomes.




**From the tree diagram, you can see that the sample space is:**

$S =$ white-white, white-black, black-white, black-black

**12G. Lesson Exercises**

**What is the sample space in each example?**

1. **A spinner with red, blue, yellow and green.**
2. **A number cube numbered 1 –6.**
3. **A bag with a blue and a red marble. One marble is drawn and then replaced.**


***Take a few minutes to check your work with a friend.***

III. Find Probabilities of Specified Outcomes Using Tree Diagrams

In the last section, you started to see how tree diagrams could be very helpful when looking for a sample space. Tree diagrams can also be helpful when finding probability.

Finding the probability of an event is a matter of finding the ratio of ***favorable outcomes*** to total outcomes. For example, the ***sample space*** for a single coin flip has **two outcomes:** heads and tails. So the probability of getting heads on any single coin flip is:

$P(\text{heads}) = \frac{favorable\ outcomes}{total\ outcomes} = \frac{1}{2}$

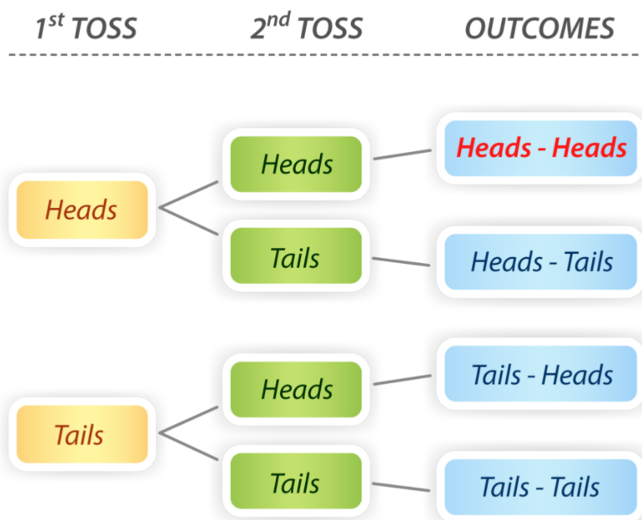
**You can see that the sample space is represented by a number in the total outcomes. For example, if you had a spinner with four colors, the colors by name would be the sample space and the number four would be the total possible outcomes.**

**What about if we flipped a coin more than one time?**

**To find the probability of a single outcome for more than one coin flip, use a tree diagram to find all possible outcomes in the sample space.**

**Then count the number of favorable outcomes within that sample space to find the probability.**

For example, to find the probability of tossing a single coin twice and getting heads both times, make a tree diagram to find all possible outcomes.




**The diagram shows there are 8 total outcomes and they are paired with first toss option and second toss option.**

**Then pick out the favorable outcome–in this case, the outcome "heads-heads" is shown in red. You could have selected any of the favorable outcomes for the probability to be accurate.**

Now write the ratio of favorable outcomes to total outcomes in the sample space.

$P(\text{heads-heads}) = \frac{favorable\ outcomes}{total\ outcomes} = \frac{1}{4}$

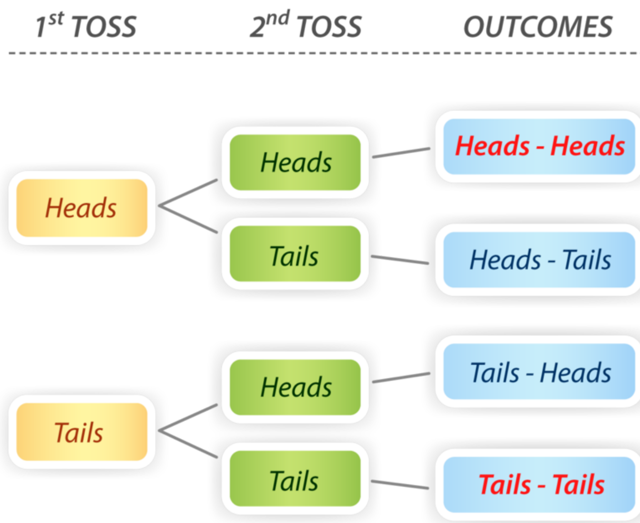
**You can see that since 1 of 4 outcomes is a favorable outcome, the probability of the coin landing on heads 2 times in a row is $\frac{1}{4}$.**

Let's look at another example.

Example

What is the probability of flipping a coin two times and getting two matching results–that is, either two heads or two tails?

**First, let's create a tree diagram to see our options.**




**Once again, just pick out the favorable outcomes on the same tree diagram. They are shown in red.**

**You can see that 2 of 4 total outcomes match.**

$P(2 \text{ heads or } 2 \text{ tails}) = \frac{favorable\ outcomes}{total\ outcomes} = \frac{2}{4} = \frac{1}{2}$

**You can see that the probability of flipping two heads or two tails is 1:2.**

## Real–Life Example Completed

***The Talent Show Outfit***

**Here is the original problem once again. Reread it and then look at the tree diagram created.**

Alicia is going to sing for the Talent Show. She is very excited and has selected a wonderful song to sing. She has been practicing with her singing teacher for weeks and is feeling very confident about her ability to do a wonderful job.

Her performance outfit is another matter. Alicia has selected a few different skirts and a few different shirts and shoes to wear. Here are her options for shirts

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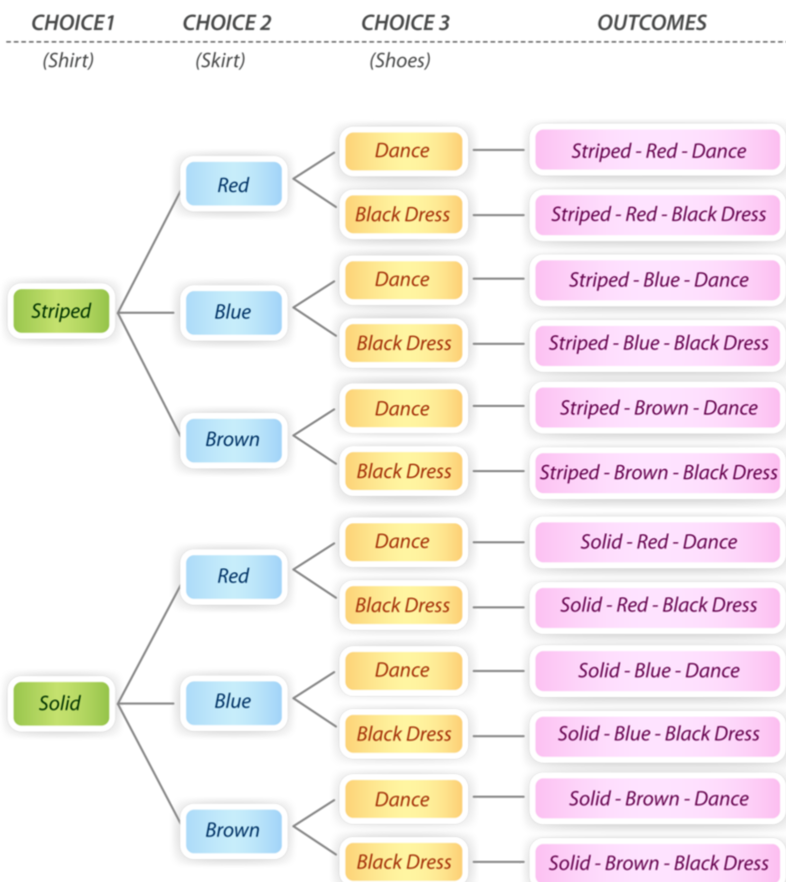
How many different outfits can Alicia create given these options?

**This is best done using a tree diagram. Alicia needs to organize her clothing options using a tree diagram. To do this, we can take each option and create a diagram to show all of the options.**




**Based on this tree diagram, you can see that Alicia has twelve possible outfits to choose from.**

## Vocabulary

Here are the vocabulary words that are found in this lesson.

**Tree Diagram**

a visual way of showing all of the possible outcomes of an experiment. Called a tree diagram because each option is drawn as a branch of a tree

**Sample Space**
The possible outcomes of an experiment

**Favorable Outcome**
the outcome that you are looking for in an experiment

**Total Outcome**
the number of options in the sample space

## Time to Practice

Directions: Use Tree Diagrams for each of the following problems.

1. The Triplex Theater has 3 different movies tonight: Bucket of Fun, Bozo the Great, and Pickle Man. Each movie has an early and late show. How many different movie choices are there?

2. Raccoon Stadium offers the following seating plans for football games:

- lower deck, middle loge, or upper bleachers
- center, side, end-zone

How many different kinds of seats can you buy?

3. Cell-Gel cell phone company offers the following choices:

- Free internet plan or Pay internet plan
- 1200, 2000, or 3000 minutes
- Premium or standard phone

How many different kinds of plans can you get?

4. Jen's soccer team is playing 4 games next week. How many different outcomes are there for the four games?

5. The e-Box laptop computer offers the following options.

- Screen: small, medium, or large
- Memory: standard 1 GB, extra 2 GB
- Colors: pearl, blue, black

List the number of different activity choices a camper can make. Use a tree diagram to list them all. Double click to check your answers.

Directions: Answer the following questions about sample spaces.

6. What is the sample space for a single toss of a number cube?

7. What is the sample space for a single flip of a coin?

8. A coin is flipped two times. List all possible outcomes for the two flips.

9. A coin is flipped three times in a row. List all possible outcomes for the three flips.

10. A bag contains 3 ping pong balls: 1 red, 1 blue, and 1 green. What is the sample space for drawing a single ball from the bag?

11. A bag contains 3 ping pong balls: 1 red, 1 blue, and 1 green. What is the sample space for drawing a single ball, returning the ball to the bag, then drawing a second ball?

12. What is the sample space for a single spin of the spinner with red, blue, yellow and green sections?

13. What is the sample space for 2 spins of the first spinner?

14. A box contains 3 socks: 1 black, 1 white, and 1 brown. What is the sample space for drawing a single sock, NOT returning the sock to the box, then drawing a second sock?

15. A box contains 3 socks: 1 black, 1 white, and 1 brown. What is the sample space for drawing all 3 socks from the box, one at a time, without returning any of the socks to the box?

16. A box contains 3 black socks. What is the sample space for drawing all 3 socks from the box, one at a time, without returning any of the socks to the box?

17. A box contains 2 black socks and 1 white sock. What is the sample space for drawing all 3 socks from the box, one at a time, without returning any of the socks to the box?

Directions: Answer each question. Use tree diagrams when necessary.

18. What is the probability that the arrow of the spinner will land on red on a single spin?

19. If the spinner is spun two times in a row, what is the probability that the arrow will land on red both times?

20. If the spinner is spun two times in a row, what is the probability that the spinner will land on the same color twice?

21. If the spinner is spun two times in a row, what is the probability that the arrow will land on red at least one time?

22. If the spinner is spun two times in a row, what is the probability that the spinner will land on a different color both times?

23. If the spinner is spun two times in a row, what is the probability that the arrow will land on blue or green at least one time?

24. Two cards, the Ace and King of hearts, are taken from a deck, shuffled, and placed face down. What is the probability that a single card chosen at random will be an Ace?

25. If one card is chosen from the 2-card stack above, then returned to the stack and a second card is chosen, what is the probability that both cards will be Kings?

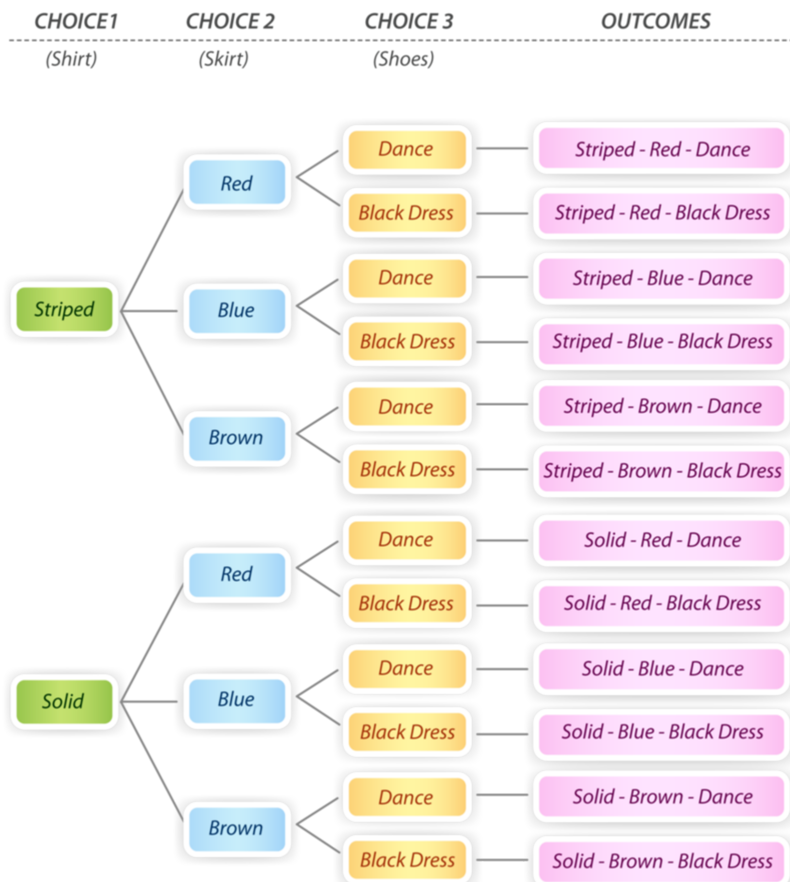
26. If one card is chosen from the 2-card stack above, then returned to the stack and a second card is chosen, what is the probability that both cards will match?

27. If one card is chosen from the 2-card stack above, then returned to the stack and a second card is chosen, what is the probability that both cards NOT match?

# 1.5 The Counting Principle

## Introduction

***Alicia's Outfit***




Remember Alicia's outfit? Well, when Alicia was working on figuring out the number of possible outfits that she could create, we used a tree diagram. A tree diagram is very useful because it provides us with a visual display of the data.

From the tree diagram, we could count all of the possible outcomes. There are 12 possible outfits for Alicia to choose from.

What about if we didn't want to draw a tree diagram? Is there another way that we could have thought about figuring out the number of outfits possible?

**This lesson is all about The Counting Principle. The Counting Principle makes figuring outcomes possible by using a mathematical method, not a tree diagram. Use this lesson to learn about The Counting Principle and we will apply it to Alicia's outfits at the end of the lesson.**

**View this video for an explanation of the Counting Principle:**

**The Counting Principle** http://www.educreations.com/lesson/view/the-counting-principle/9590690/?ref=appemail

***What You Will Learn***

In this lesson, you will learn how to do the following things:

- Recognize the number of possible outcomes of a series of events as the product of the number of possible outcomes for each event.
- Use the Counting Principle to find all possible outcomes of a series of events involving two or more choices or results.
- Find probabilities of specified outcomes using the Counting Principle.

***Teaching Time***

I. **Recognize the Number of Possible Outcomes of a Series of Events as the Product of the Number of Possible Outcomes for Each Event**

In the last lesson, you learned about tree diagrams. Tree diagrams provide you with a visual way of seeing all of the possible outcomes for a set of particular events.

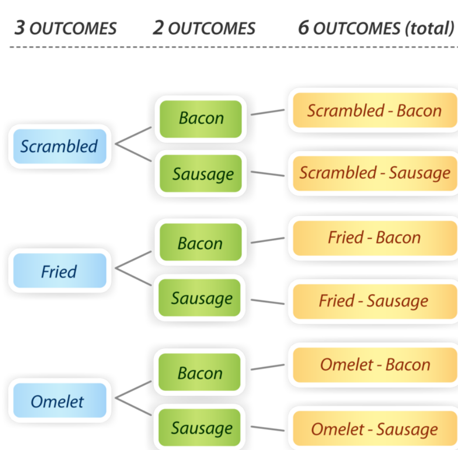
**What if there was a simpler way?**

Sometimes, you don't want to have to draw a tree diagram to figure out all of the possible outcomes for a series of events. When this happens, you can use another principle to figure out the possible outcomes.

Let's look at an example.

Example

Molly's All Star Farm Breakfast features 3 choices of eggs–scrambled, fried, or omelet–plus a choice of bacon or sausage. You can use a tree diagram to find that there are 6 different choices, or outcomes, for the breakfast.



**What if we wanted to look at this in another way?**

We could look at the number of possible breakfast options in terms of outcomes.

**For the first choice there are 3 different outcomes. For the second choice there are 2 different outcomes.**

3 outcomes $\cdot$ 2 outcomes $=$ 6 outcomes


**Yes they are. That is because this is an example of a new principle for figuring out the total possible outcomes of a series of events. We call it the *Counting Principle.***

**Counting Principle:** The number of choices or outcomes for two independent events, $A$ and $B$, taken together, is the product of the total number of outcomes for each event.

Total outcomes for $A$ and $B = ($number of outcomes for $A) \cdot ($number of outcomes for $B)$

II. <u>**Use the Counting Principle to Find all Possible Outcomes of a Series of Events Involving Two or More Choices or Results**</u>

Now that you know what the Counting Principle is, you can practice using it. The Counting Principle will work for 2, 3 even 4 or more different events. Just follow the procedure of using multiplication to find the number of possible outcomes.

Let's apply the Counting Principle to an example.

Example

For buying gum you have the following choices:

- 3 flavor choices–spearmint, peppermint, cinnamon
- 2 sugarless choices–sugarless or non-sugarless
- 2 bubble choices–bubble gum or regular

**To find the number of gum choices you have you could make a tree diagram, or you could simply use the Counting Principle:**

3 choices $\cdot$ 2 choices $\cdot$ 2 choices $=$ 12 choices

To check the answer, you can write out all of the possible options for gum:

| | | |
|---|---|---|
| spear-sugarless-bubble | pepper-sugarless-bubble | cinnamon-sugarless-bubble |
| spear-sugarless-regular | pepper-sugarless-regular | cinnamon-sugarless-regular |
| spear-non-bubble | pepper-non-bubble | cinnamon-non-bubble |
| spear-non-regular | pepper-non-regular | cinnamon-non-regular |

**You can see that the Counting Principle worked out fine for the solution.**

Example

You're buying a sweater and have the following choices.

- 5 different color choices–black, yellow, blue, red, green
- 3 different material choices–wool, cotton, fleece
- 4 different style choices–v-neck, crew, button-down, turtle

**To find the number of sweater choices you have, you could make a tree diagram, or you could simply use the Counting Principle:**

5 choices $\cdot$ 3 choices $\cdot$ 4 choices $=$ 60 choices

**12H. Lesson Exercises**

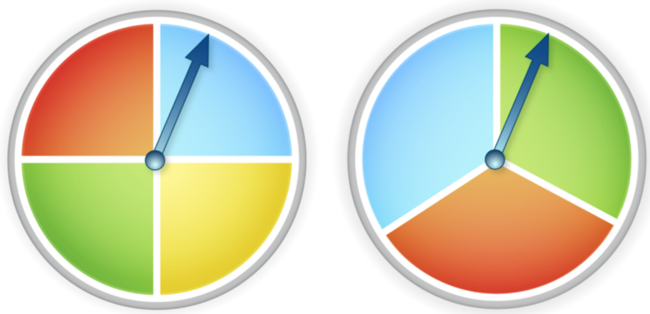
**Use the Counting Principle to count outcomes.**

1. **Omar is buying a skateboard. He has 5 different skateboard decks to choose from and 4 different wheel choices. How many different skateboard choices does Omar have?**

2. **Ice Stone ice cream shop has 3 different sundae sizes: baby, large, and grand. You can choose from 6 different ice cream flavors and add 4 different toppings. How many sundae choices are there?**
3. **Gina tosses a number cube 2 times. How many different outcomes are there?**


***Take a few minutes to check your answers with a partner.***

III. **Find Probabilities of Specified Outcomes Using the Counting Principle**



You can use the Counting Principle to help find probabilities of events. For example, suppose you wanted to know the probability of the arrow landing on the same color on both spinners. Keep in mind that for any probability you can use this ratio.

$P(\text{event}) = \frac{\textit{favorable outcomes}}{\textit{total outcomes}}$

Here, you can use the Counting Principle to find the number of total outcomes for the two spins with the two spinners. There are four outcomes on one spinner and three outcomes on the other spinner.

$$\begin{aligned}\text{Total outcomes} &= 4 \text{ outcomes} \cdot 3 \text{ outcomes}\\ &= 12 \text{ outcomes}\end{aligned}$$

Now list those 12 outcomes and mark the outcomes that are the same color for both spins.

| | | | |
|---|---|---|---|
| red-red | blue-red | yellow-red | green-red |
| red-blue | blue-blue | yellow-blue | green-blue |
| red-green | blue-green | yellow-green | green-green |

Since there are 3 outcomes that have the same color for both spins:

$P(\text{same}) = \frac{3}{12} = \frac{1}{4}$

**The probability of both spinners landing on the same color is $\frac{1}{4}$.**

You can also apply the Counting Principle to a variety of different probability problems.

Example

Anna flips a coin 3 times in a row. What is the probability that she will get heads all 3 times?

***Step 1:*** Rather than draw a tree diagram, to find the number of total outcomes you can simply multiply the number of outcomes for each flip

$$\begin{aligned}\text{Total outcomes} &= 2 \text{ outcomes} \cdot 2 \text{ outcomes} \cdot 2 \text{ outcomes}\\ &= 8 \text{ outcomes}\end{aligned}$$

***Step 2:*** Now list all 8 outcomes and find the number of ways Anna can get heads all 3 times. Clearly, there is only one arrangement in which all 3 flips result in heads.

| | |
|---|---|
| heads-heads-heads | tails-heads-heads |
| heads-heads-tails | tails-heads-tails |
| heads-tails-heads | tails-tails-heads |
| heads-tails-tails | tails-tails-tails |

***Step 3:*** Find the ratio of favorable outcomes to total outcomes:

$P(\text{red-red-red}) = \frac{1}{8}$

**12I. Lesson Exercises**

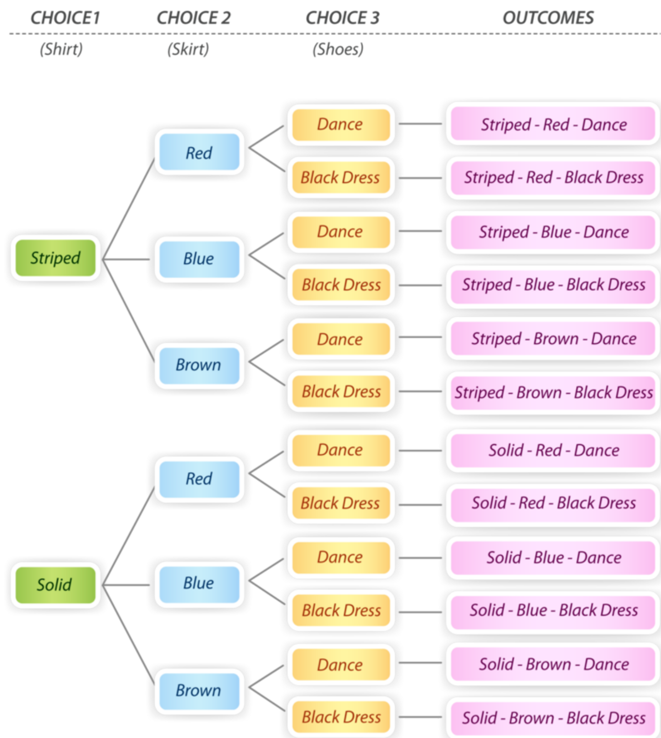
**Use the Counting Principle to find each probability.**

1. **Abra flips a coin 2 times. What is the probability that both flips will match?**
2. **Abra flips a coin 2 times. What is the probability that both flips will NOT match?**
3. **Abra flips a coin 3 times. What is the probability that all 3 flips will match?**


***Check each answer with a friend.***

## Real–Life Example Completed

***Alicia's Outfit***




**Here is the original problem once again. Reread it and then we will look at applying The Counting Principle to this problem.**

Remember Alicia's outfit? Well, when Alicia was working on figuring out the number of possible outfits that she could create, we used a tree diagram. A tree diagram is very useful because it provides us with a visual display of the data.

From the tree diagram, we could count all of the possible outcomes. There are 12 possible outfits for Alicia to choose from.

What about if we didn't want to draw a tree diagram? Is there another way that we could have thought about figuring out the number of outfits possible?

**Think back to the lesson about The Counting Principle.**

**Total Outcomes = (Number of outcomes)(Number of outcomes)**

**With Alicia's problem, there are three possible numbers of outcomes. We have the shirts that she selected as options, there are two of them. We have the skirts that she selected as options, there are three of those. We have the shoes that she selected as options, there are two of those.**

$3 \times 2 \times 2 = 12$ possible outfits

**You can see that we ended up with the same answer as we did with the tree diagram. The Counting Principle definitely works.**

## Vocabulary

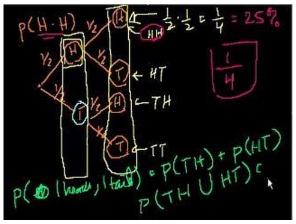
Here are the vocabulary words that are found in this lesson.

**Probability**
the ratio of favorable outcomes to total possible outcomes.

**The Counting Principle**

the product of the outcomes of a series of events gives the total number of outcomes.

### Technology Integration



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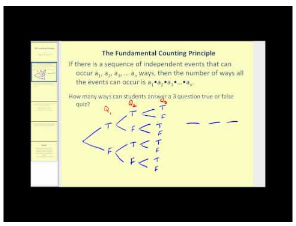
URL: http://www.ck12.org/flx/render/embeddedobject/1403

Khan Academy, Probability Part 2




MEDIA
Click image to the left or use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/5461

James Sousa, The Counting Principle

Other Videos:

1. http://www.5min.com/Video/Application-of-the-Fundamental-Counting-Principle-of-Probability-275652517 – This video shows how to use the Counting Principle on a variety of different problems from a worksheet.

### Time to Practice

Directions: Use the Counting Principle to solve each problem.

1. The Cubs have 3 games left to play this year. How many different outcomes can there be for the three games?

2. Svetlana tosses a coin 4 times in a row. How many outcomes are there for the 4 tosses?

3. For a new tennis racquet, Danny can choose from 8 different brands, 3 different head sizes, and 4 different grip sizes. How many different racquet choices does Danny have?

4. Gina tosses a number cube 3 times. How many different outcomes are possible?

5. Gina tosses a number cube. Buster flips a coin. How many different outcomes are possible for the two events?

6. Buster flips a coin. Daoud chooses a card from a deck of 52 cards. How many different outcomes are possible for the two events?

7. Rex spins a spinner that has red, blue, and yellow sections two times. How many different outcomes are possible?

8. Daoud chooses a card from a deck of 52 cards, replaces the card in the deck, then chooses a second card. How many different outcomes are possible?

9. Patsy's Pizza features 3 different pizza types, 14 different toppings, and 2 different sizes. How many different pizzas can you order?

10. Spud has a 2-letter password for his computer using letters only. If there are 26 different letters in the alphabet, how many different passwords are possible?

11. Doreen has a 3-digit password for her computer using digits only. If there are 10 different digits (including zero), how many different passwords are possible?

12. Sebastian has a 3-letter password for his computer using vowels (A, E, O, I, U) only. How many different passwords are possible?

Directions: Find probabilities using the Counting Principle.

13. Abra flips a coin 3 times. What is the probability that all 3 flips will NOT match?

14. Abra flips a coin 3 times. What is the probability that heads will come up exactly 2 times?

15. Billy spins the spinner twice. What is the probability that blue will come up both times?

16. Billy spins the spinner twice. What is the probability that blue will come up at least one time?

17. Billy spins the spinner twice. What is the probability that blue will come up exactly one time?

18. Billy spins the spinner twice. What is the probability that both spins will match?

19. Cindy tosses a number cube two times. What is the probability that both tosses will match?

20. Cindy tosses a number cube two times. What is the probability that both tosses will NOT match?

21. Cindy tosses a number cube two times. What is the probability that the sum of the two tosses will be 7?

22. Cindy tosses a number cube two times. What is the probability that the sum of the two tosses will be greater than 7?

# 1.6 Permutations

## View this video on Permutations:

**Permutations** http://www.educreations.com/lesson/view/permutations/9963827/?ref=appemail

**To do Permutations on the Graphing Calculator:**

**Example: How many possible permutations of 2 cards can be chosen from a deck of 5 cards?**

• Input 5. • Press [MATH], arrow left to highlight PRB, then press [2] to select the nPr function.

• Input 2 and press [ENTER]. There are 20 possible permutations of choosing 2 cards from a deck of 5 cards.

## Introduction

***The Order of Entertainment***

Greg and Joanna are in charge of creating an order for the Talent Show. There are 6 sixth graders, 10 seventh graders and 6 eighth graders who have entered the contest.

The order that the students perform in makes a difference, and the sixth graders will all perform together. Then the seventh graders will perform and finally the eighth graders will perform.

Greg and Joanna begin with the sixth graders. Since there are six sixth graders who are performing and order does make a difference, how many different arrangements of the order are possible?

"I don't have a clue how to figure this out," Joanna says to Greg.

"I think I do, let me get a piece of paper."

**While Greg gets a piece of paper, it is time for you to think about this problem. Solving it has to do with something called a "permutation." This lesson is all about permutations and how to figure them out.**

**Pay attention and at the end of the lesson you will know how to help Greg and Joanna figure out the order of sixth graders.**

***What You Will Learn***

By the end of this lesson, you will know how to accomplish the following skills tests:

- Recognize permutations as arrangements in which order is important.
- Count all permutations of $n$ objects or events.
- Count permutations of $n$ objects taken $r$ at a time.
- Evaluate permutations using permutation notation.

***Teaching Time***

I. **Recognize Permutations as Arrangements in which Order is Important**

You learned in that last lesson about combinations. You can make all kinds of combinations. Let's say that you are making a pizza with pepperoni, mushrooms and peppers. It doesn't matter which order you put the toppings on the pizza. You will still have the same pizza.

Sometimes, order does make a difference. **When you arrange different objects or events and order is important, we call each arrangement a *permutation.*** This lesson is all about permutations.

**Think about spelling.**

Consider the word CAT. Clearly, order is important when you spell a word. You can write all of the correct letters, but if you don't put them in the correct order, you don't spell CAT. For example, here are some orders of C, A, and T that *don't* spell CAT;

Incorrect orders: $ACT, ATC, CTA, TAC, TCA$
Correct orders: $CAT$

This is an example where a permutation makes a difference. We can also use permutations to solve problems. To use permutations to solve problems, you need to be able to identify the problems in which order, or the arrangement of items, matters.

Let's look at an example.

Example

Tomás wants to know how many 3-digit numbers he can write using the digits 7, 8, and 9 without repeating any of the digits. Does order matter for this problem?

***Step 1:*** Write out a single order.

789

***Step 2:*** Now **rearrange** the order. Did you change the outcome? If so, then order matters.

798 This is different from the original number

**Each arrangement of digits is a different permutation.**

Example

The softball coach needs to determine how many different batting lineups she can make out of her first three batters, Able, Baker, and Chan. Does order matter for this problem?

***Step 1:*** Write out a single lineup.

Able, Baker, Chan

***Step 2:*** Now rearrange the lineup. Did you change the outcome? If so, then order matters.

Able, Chan, Baker–this order is different from the original

**Each arrangement of batters is a different permutation.**

**12J. Lesson Exercises**

**Write whether or not order is important for each scenario and why.**

1. **How many 3-letter words can Brenda write using the letters $E, T, A$ without repeating any letters?**
2. **Five different cars entered the race, each painted one of the following colors: red, orange, blue, white, purple? In how many different ways can the cars finish the race?**
3. **At the breakfast buffet you can take any three of the following: eggs, pancakes, potatoes, cereal, waffles. How many different 3-item breakfasts can you get?**


***Check your answers with a partner.***

II. **Count all Permutations of $n$ Objects or Events**

Once you decide that the order does matter, you know that you are working with a permutation. It is helpful to know how to calculate permutations.

Let's look at an example to see how to do this.

Example

The softball coach needs to determine how many different batting lineups she can make out of her first three batters, Able, Baker, and Chan. How many different batting orders are there?

**One way to look at this problem is as the product of 3 different choices. For choice 1 you can select any of the three batters, Able, Baker, or Chan.**

Choice 1 $\times$ Choice 2 $\times$ Choice 3 $=$ Possible Number of Choices

3 is the number of choices in Choice 1 because you have 3 batters to choose from.

2 is the number of choices in Choice 2 because you selected one batter for Choice 1 leaving 2 choices.

1 is the number of choices in Choice 3 because that is all that is left.

**Here is the answer:**

$3 \times 2 \times 1 = 6$

**There are six possible choices.**

Using the Counting Principle you can multiply the three choices together to get the total number of choices, or permutations, as 6. Here are the 6 different batting lineups.

| | | |
|---|---|---|
| Able-Baker-Chan | Baker-Able-Chan | Chan-Able-Baker |
| Able-Chan-Baker | Baker-Chan-Able | Chan-Baker-Able |

Notice that order is important here. Each of the 6 choices, or permutations, is a unique and different batting order. For example, Able-Baker-Chan is not the same lineup as Able-Chan-Baker.

What happens when you increase the number of players in your lineups by adding Davis? How many different lineups are there now? Starting over, you can now see that there are 4 choices for the first batter, followed by 3 choices, 2 choices, and 1 choice.

$4 \times 3 \times 2 \times 1 = 24$ possible choices

Let's look at another example.

Example

How many different arrangements of the letters $A, B, C, D$, and $E$ can you make without repeating any of the letters?

**This is very much like the Able, Baker, Chan, Davis problem only it adds a fifth item.** Notice how this extra item increases the total by a huge amount. In fact, there are exactly 5 times as many permutations of 5 items than there were of 4 items above.

$$\underset{\text{choice 1}}{\boxed{5}} \cdot \underset{\text{choice 2}}{\boxed{4}} \cdot \underset{\text{choice 3}}{\boxed{3}} \cdot \underset{\text{choice 4}}{\boxed{2}} \cdot \underset{\text{choice 5}}{\boxed{1}} = \underset{\text{total choices}}{\boxed{120}}$$

You can see how the product of all of the choices was figured out in this problem.

III. **Count Permutations of $n$ Objects Taken $r$ at a Time**

In the last section, you learned how to count permutations of a certain amount of choices where all of the choices were used each time. For example, when you completed the batter line up of three batters in order, you could count the permutations like this.

$3 \times 2 \times 1 = 6$ possible permutations

**What happens if you were to have four players, but you only wanted to put three in the line up?**

This changes the way that we count permutations.

**To accomplish this task, you start counting at 4 and then find the product of the next two values as well.**

$4 \times 3 \times 2 = 24$ permutations

***Notice that we don't include the 1 because there are four batters taken three at a time. Since we are only using 3 of the 4 options, we only multiply the first 3 of the 4 counts.***

Let's look at another example.

Example

What will happen if Elvis joins the team? Now, with 5 players to choose from, how many different 3-player batting orders are there?

**You still have 3 different choices to make, but this time you have 5 different players for your first choice, 4 players for your second choice, and 3 players for your third choice.**

$5 \times 4 \times 3 = 60$ options

**You can use this method to find any permutation count, no matter how many options there are.**

Example

Taken 4 at a time, how many different arrangements of the letters $A, B, C, D, E$, and $F$ are there?

**First, notice that there are six possible letter choices to work with. You want to take them four at a time.**

$6 \times 5 \times 4 \times 3 = 360$ possible options

***Notice that you don't have to see all 360 options to know that your answer is accurate! If you followed the method of counting permutations, then your answer is correct.***

IV. **Evaluate Permutations Using Permutation Notation**

In the last section, you figured out permutations by arranging numbers. Sometimes you used boxes to organize the numbers and sometimes you just wrote out the multiplication problem. **We can use *permutation notation* to help us know when we are working with a permutation.**

**What is permutation notation?**

Permutation notation involves something called a ***factorial*** **. A *factorial* is a way of writing a number to show that we are going to be looking for the product of a series of numbers.**

**The symbol for a factorial is an exclamation sign.**

**Here are some examples of factorials.**

$$\begin{aligned}
8! &= 8 - \text{factorial} \\
11! &= 11 - \text{factorial} \\
29! &= 29 - \text{factorial} \\
2! &= 2 - \text{factorial, and so on}
\end{aligned}$$

**To compute factorials, simply rewrite the number before the exclamation point and all of the whole numbers that come before it.**

$$\begin{aligned}
4! &= 4 \cdot 3 \cdot 2 \cdot 1 \\
7! &= 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\
2! &= 2 \cdot 1 \\
11! &= 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1
\end{aligned}$$

**How do you calculate the values of factorial numbers? To find out, simply multiply.**

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$
$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$
$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$


**Yes. That is all that you need to know. As long as you remember how to find the product of a factorial, you will always know a short cut for permutations!**

**Now that you've learned how to work with factorials, you are ready see how to use them to calculate permutations.** Suppose you have 5 letters – $A, B, C, D$, and $E$. You want to know how many permutations there are if you take 3 letters at a time and find all arrangements of them. In ***permutation notation*** you write this as:

${}_5P_3 \Longleftarrow$ 5 items taken 3 at a time

In general, permutations are written as:

${}_nP_r \Longleftarrow n$ items taken $r$ at a time

To compute ${}_nP_r$ you write:

${}_nP_r = \frac{n!}{(n-r)!} = \frac{\Longleftarrow \text{total items!}}{\Longleftarrow (\text{total items-items taken at a time})!}$

To compute ${}_5P_3$ just fill in the numbers:

$${}_5P_3 = \frac{5!}{(5-3)!} = \frac{\Longleftarrow \text{ total items!}}{\Longleftarrow \text{ (total items-items taken at a time)!}}$$
$${}_5P_3 = \frac{5(4)(3)(2)(1)}{2(1)} = \frac{120}{2} = 60$$

**There are 60 possible permutations with this example.**

**You can use this formula anytime that you are looking to figure out permutations!**

## Real–Life Example Completed

***The Order of Entertainment***

**Here is the original problem once again. Reread it and then work through figuring out the permutation.**

Greg and Joanna are in charge of creating an order for the Talent Show. There are 6 sixth graders, 10 seventh graders and 6 eighth graders who have entered the contest.

The order that the students perform in makes a difference, and the sixth graders will all perform together. Then the seventh graders will perform and finally the eighth graders will perform.

Greg and Joanna begin with the sixth graders. Since there are six sixth graders who are performing and order does make a difference, how many different arrangements of the order are possible?

"I don't have a clue how to figure this out," Joanna says to Greg.

"I think I do, let me get a piece of paper."

**Greg takes out a piece of paper and writes this on it.**

**6 sixth graders**

$6 \times 5 \times 4 \times 3 \times 2 \times 1 = the\ number\ of\ possible\ arrangements$

**"You see, it makes a difference, so we can use a factorial," Greg explains. "Now we will know how many possible arrangements of sixth graders there are."**

$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

**There are 720 possible combinations. Greg and Joanna probably need to narrow this down a little further because that is a lot of possible arrangements. They decide to make singers one category. That will help them with possible combinations.**

### Vocabulary

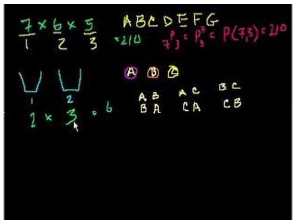
Here are the vocabulary words that are found in this lesson.

**Permutation**
a combination where the order matters.

**Permutation Notation**
using the factorial symbol to show a permutation

**Factorial**
a short-cut for permutations. An exclamation point next to a number is the symbol for factorial. It means "Count down from this number to 1, then multiply each of those counting numbers together into a total."

### Technology Integration



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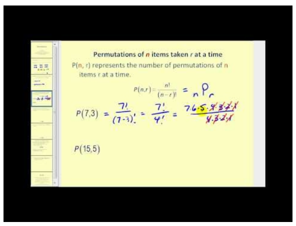
Khan Academy, Permutations



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James Sousa, Permutations

### Time to Practice

Directions: Decide whether or not order matters for each of the following scenarios. Briefly explain your reasoning.

1. Doug is going to use the following 5 letters to create his new 3-letter computer password: B, F, G, L, and T. How many different passwords can he create if he doesn't repeat any letters?

2. Violin players in the orchestra include Jerry, Kerry, Barry, Mary, Sherry, Harry, Terri, and Perry. How many different 3-person trios can you make?

3. The 3 different numbers for Arun's lock are 14, 35, and 20. How many different combinations must Arun try before he'll be sure he can open his lock?

4. Mr. Chen has decided that he's going to give Nikki, Mickey, and Hickey awards for the essay contest. What he doesn't know is who will get 1st prize, 2nd prize, and 3rd prize. How many different ways can Mr. Chen give out the prizes?

5. Five candidates are running for 2 student senate seats: Bo, Jo, Mo, Zo, and Ro. How many different pairs of senators can there be?

6. Five skaters are competing in the County Championship Finals: Miller, Diller, Hiller, Giller, and Stiller. How many different ways can they finish first, second, third, fourth, and fifth?

Directions: Find the number of permutations for each problem.

7. How many 3-letter sequences can Brenda write using the letters A, B, and C without repeating any of the letters?

8. How many 4-digit numbers can Brenda write using the digits 1, 3, 5, 7 without repeating any of the digits?

9. Doug, Eileen, Francesca, and Garth all entered the swimming race. In how many different orders can the four racers finish?

10. Miguel is serving soup, salad, pasta, and fish for dinner. In how many different orders can he serve the four dishes?

11. Mike has 4 different playing cards: Ace, King, Jack, and Ten. How many different 4-card arrangements can he make?

12. Marlena strung 5 charms on a bracelet–a star, a fish, a diamond, a moon, and a baby shoe. Into how many different orders can she arrange the 5 charms?

13. Michelle forgot her 6-letter computer password. She knows she used the letters H, I, J, K, L, and M in the password and that she didn't repeat any of the letters. How many different passwords must she try before she is sure to hit the correct one?

14. Seven skaters are competing in the County Championship Finals. In how many different orders can they finish first, second, third, fourth, fifth, sixth, and seventh?

Directions: Count all of the permutations.

15. Three marbles–red, blue, yellow,–are in a jar. In how many different orders can you pull two of the marbles out of the jar (without replacing either of the marbles in the jar)?

16. A green marble is added to the jar above, giving red, blue, yellow and green marbles. In how many different orders can you pull three of the marbles out of the jar (without replacing any of the marbles in the jar)?

17. In a jar with 4 marbles–red, blue, yellow, and green–how many different orders will you have if you pull just 2 marbles from the jar (without replacing either of the marbles in the jar)?

18. In a jar with 5 marbles–red, blue, yellow, green, and white–how many different orders are possible if you pull 3 marbles from the jar?

19. How many 4-digit arrangements can you make of the digits 1, 2, 3, 4, 5 if you don't repeat any digit?

20. A TV channel has 6 different 1-hour shows to fill 3 hours of time for Thursday night. How many different program lineups can the channel present?

21. Seven ski racers compete in the finals of the slalom event. In how many different orders can the top 3 skiers finish?

22. Seven ski racers compete in the finals of the downhill event. In how many different orders can the top 4 skiers finish?

Directions: Solve each factorial.

23. 5!

24. 3!

25. 6!

26. 4!

Directions: Use permutation notation and the formula to find each permutation.

27. ${}_4P_2$

28. ${}_4P_3$

29. ${}_5P_4$

30. ${}_6P_3$

# 1.7 Combinations

## View this video on Combinations:

**Combinations** http://www.educreations.com/lesson/view/combinations-vs-permutations/10172960/?ref=appemail

To do Combinations on the Graphing Calculator:

**Example: How many possible combinations of 2 cards can be chosen from a deck of 5 cards?**

• Input 5. • Press [MATH], arrow left to highlight PRB, then press [3] to select the nCr function.

• Input 2 and press [ENTER]. There are 10 possible combinations of choosing 2 cards from a deck of 5 cards

## Introduction

***Decorating the Stage***

The decorating committee is getting the stage ready for the Talent Show. There was a bunch of different decorating supplies ordered, and the students on the committee are working on figuring out the best way to decorate the stage.

They have four different colors of streamers to use to decorate.

Red

Blue

Green

Yellow

"I think four is too many colors. How about if we choose three of the four colors to decorate with?" Keith asks the group.

"I like that idea," Sara chimes in. "How many ways can we decorate the stage if we do that?"

The group begins to figure this out on a piece of paper.

**Combinations are arrangements where order does not make a difference. The decorating committee is selecting three colors from the possible four options. Therefore, the order of the colors doesn't matter.**

**Combinations are the way to solve this problem. Look at the information in this lesson to learn how to figure out the possible combinations.**

***What You Will Learn***

In this lesson you will learn how to:

- Recognize combinations as arrangements in which order is not important.
- Count all combinations of $n$ objects or events
- Count combinations of $n$ objects taken $r$ at a time
- Evaluate combinations using combination notation.

***Teaching Time***

I. **Recognize Combinations as Arrangements in Which Order is Not Important**

In the last section, you saw that *order* is important for some groups of items but not important for others. For example, consider a list of three words: HOPS, SHOP, and POSH.

- For the spelling of each individual word, order is important. The words HOPS, SHOP, and POSH all use the same letters, but spell out very different words.
- For the list itself, order is not important. Whether the words are presented in one order–such as HOPS, SHOP, POSH, or another order, such as SHOP, POSH, HOPS, or a third order, such as POSH, HOPS, SHOP–makes no difference. As long as the list includes all 3 words, the order of the 3 words doesn't matter.

**A *combination* is a collection of items in which order, or how the items are arranged, is not important.** The collection of one order of the items is not functionally different than any other order.

Combinations and permutations are related. To solve problems in which order matters, you use ***permutations.*** To solve problems in which order does NOT matter, use ***combinations.***

Let's look at an example.

Example

The winning 3-digit lottery numbers are drawn from a drum as 641, 224, and 806. Does order matter in the way the three winning numbers are drawn?

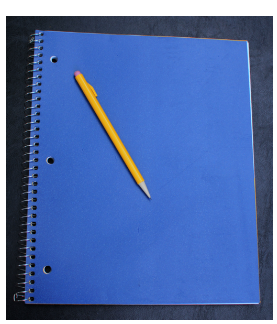
***Step 1:*** Write out a single order.

641, 224, 806

***Step 2:*** Now **rearrange** the order. Did you change the outcome? If so, then order matters.

$224, 806, 641 \Longleftarrow$ different order, same 3 winning numbers

**Order does NOT matter for this problem. Use combinations.**



***Write the difference between combinations and permutations down in your notebook.***

Example

A bag has 4 marbles: red, blue, yellow, and green. In how many different ways can you reach into the bag and draw out 1 marble, then return the marble to the bag and draw out a second marble?

***Step 1:*** Write out a single order.

red, blue

***Step 2:*** Now **rearrange** the order. Did you change the outcome? If so, then order matters.

blue, red $\Longleftarrow$ different order, meaning is DIFFERENT

**Order DOES matter for this problem. Use permutations.**

**12K. Lesson Exercises**

**Write whether you would use combinations or permutations for each example.**

1. **Cesar the dog-walker has 5 dogs but only 3 leashes. How many different ways can Cesar take a walk with groups of 3 dogs at once?**
2. **Five different horses entered the Kentucky Derby. In how many different ways can the horses finish the race?**
3. **How many different 5-player teams can you choose from a total of 8 basketball players?**


***Take a few minutes to check your answers with a friend.***

II. **Count All Combinations of *n* Objects or Events**

**Once you figure out if you are going to be using permutations or combinations, it is necessary to count the combinations.**

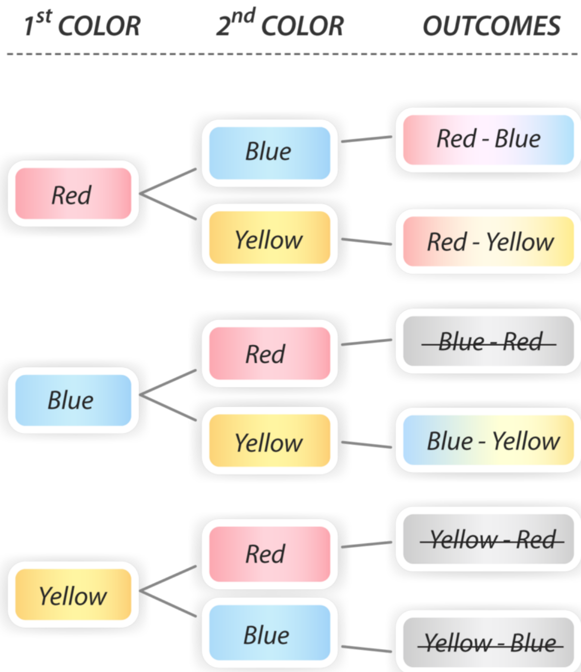
There are several different ways to count combinations. When counting, try to keep the following in mind:

- Go one by one through the items. Don't stop your list until you've covered every possible link of one item to all other items.
- Keep in mind that order doesn't matter. For combinations, there no difference between $AB$ and $BA$. So if both $AB$ and $BA$ are on your list, cross one of the choices off your list.
- Check your list for repeats. If you accidentally listed a combination more than once, cross the extra listings off your list.

Example

James needs to choose a 2-color combination for his intramural team t-shirts. How many different 2-color combinations can James make out of red, blue, and yellow?

One way to find the number of combinations is to make a tree diagram. Here, if red is chosen as one color, that leaves only blue and yellow for the second color.




The diagram shows all 6 *permutations* of the 3 colors. But wait–since we are counting **COMBINATIONS** here order doesn't matter.

So in this tree diagram we will cross out all outcomes that are repeats. For example, the first red-blue is no different from blue-red, so we'll cross out blue-red.

***In all, there are 3 combinations that are not repeats.***

**This method of making a tree diagram and crossing out repeats is reliable, but it is not the only way to find combinations.**

Let's look at another example.

Example

James has added a fourth color, green, to choose from in selecting a 2-color combination for his intramural team. How many different 2-color combinations can James make out of red, blue, yellow, and green?

***Step 1:*** Write the choices. Match the first choice, red, with the second, blue. Add the combination, red-blue, to your list. Match the other choices in turn. Add the combinations to your list.

red, blue, yellow, green

list

red-blue

red-yellow

red-green

***Step 2:*** Now move to the second choice, blue. Match blue up with every possible partner other than red, since we already included all of the combinations involving red. Add the combinations to your list.

red, blue, yellow, green

list

| | |
|---|---|
| red-blue | blue-yellow |
| red-yellow | blue-green |
| red-green | |

***Step 3:*** Now move to the third choice, yellow. There is only one new combination left to match it with. Add the combination to your list.

red, blue, yellow, green

list

| | | |
|---|---|---|
| red-blue | blue-yellow | yellow-green |
| red-yellow | blue-green | |
| red-green | | |

**Your list is now complete. There are 6 combinations.**

II. **Count All Combinations of *n* Objects Taken *r* at a Time**

Sometimes, you won't want to use all of the possible options in the combination. Think about it as if you have 16 flavors of ice cream, but you only want to use three flavors at a time. This is an example where there are 16 flavors to work with, but you can only use three at a time. With an example like this one, you are looking for combinations of object where only a certain number of them are used in any one combination.

**This happens a lot with teams.** Let's look at an example.

Example

How many different 2-player soccer teams can Jean, Dean, Francine, Lurleen, and Doreen form?

***Step 1:*** Start with Jean. Add all combinations that begin with Jean to your list.

| Combination | List |
|---|---|
| Jean, Dean, Francine, Lurleen, Doreen | Jean-Dean |
| Jean, Dean, Francine, Lurleen, Doreen | Jean-Francine |
| Jean, Dean, Francine, Lurleen, Doreen | Jean-Lurleen |
| Jean, Dean, Francine, Lurleen, Doreen | Jean-Doreen |

***Step 2:*** You've covered all combinations that begin with Jean. Now go through all combinations that begin with Dean, Francine, and Lurleen.

| Combination | List |
|---|---|
| Jean, Dean, Francine, Lurleen, Doreen | Jean-Dean |
| Jean, Dean, Francine, Lurleen, Doreen | Jean-Francine |
| Jean, Dean, Francine, Lurleen, Doreen | Jean-Lurleen |
| Jean, Dean, Francine, Lurleen, Doreen | Jean-Doreen |
| Jean, Dean, Francine, Lurleen, Doreen | Dean-Francine |
| Jean, Dean, Francine, Lurleen, Doreen | Dean-Lurleen |
| Jean, Dean, Francine, Lurleen, Doreen | Dean-Doreen |
| Jean, Dean, Francine, Lurleen, Doreen | Francine-Lurleen |
| Jean, Dean, Francine, Lurleen, Doreen | Francine-Doreen |
| Jean, Dean, Francine, Lurleen, Doreen | Lurleen-Doreen |

**Your list is now complete. There are 10 combinations.**

Example

How many different **3-player** soccer teams can Jean, Dean, Francine, Lurleen, and Doreen form?

Use the process above to go through all of the combinations.

| Combination | List |
|---|---|
| Jean, Dean, Francine, Lurleen, Doreen | Jean-Dean-Francine |
| Jean, Dean, Francine, Lurleen, Doreen | Jean-Dean-Lurleen |
| Jean, Dean, Francine, Lurleen, Doreen | Jean-Dean-Doreen |
| Jean, Dean, Francine, Lurleen, Doreen | Jean-Francine-Lurleen |
| Jean, Dean, Francine, Lurleen, Doreen | Jean-Francine-Doreen |
| Jean, Dean, Francine, Lurleen, Doreen | Jean-Lurleen-Doreen |
| Jean, Dean, Francine, Lurleen, Doreen | Dean, Francine-Lurleen |
| Jean, Dean, Francine, Lurleen, Doreen | Dean-Francine-Doreen |
| Jean, Dean, Francine, Lurleen, Doreen | Dean-Lurleen-Doreen |
| Jean, Dean, Francine, Lurleen, Doreen | Francine-Lurleen-Doreen |

**Your list is now complete. There are 10 combinations.**

**Try a few of these on your own.**

## 12L. Lesson Exercises

1. **On Monday Cesar the dog-walker has 3 dogs–Looie, Huey, and Dewey-but only 2 leashes. How many different ways can Cesar take a walk with 2 dogs? List the ways.**
2. **On Tuesday Cesar has 4 dogs-Looie, Huey, Dewey, and Stewie–but only 2 leashes. How many different ways can Cesar take a walk with 2 dogs? List the ways.**
3. **On Wednesday Cesar has 4 dogs-Looie, Huey, Dewey, and Stewie–but now has 3 leashes. How many different ways can Cesar take a walk with 3 dogs? List the ways.**


***Take a few minutes to discuss your findings with a partner. Share your method of finding all of the possible combinations.***

IV. **Evaluate Combinations Using Combination Notation**

We can use a formula to help us to calculate combinations. This is very similar to the work that you did in the last section with factorials and permutations.

Example

Suppose you have 5 marbles in a bag–red, blue, yellow, green, and white. You want to know how many combinations there are if you take 3 marbles out of the bag all at the same time. In combination notation you write this as:

${}_5C_3 \Longleftarrow$ 5 items taken 3 at a time

In general, combinations are written as:

${}_nC_r \Longleftarrow n$ items taken $r$ at a time

To compute ${}_nC_r$ use the formula:

${}_nC_r = \frac{n!}{r!(n-r)!}$

**This may seem a bit confusing, but it isn't. Notice that the factorial symbol is used with the number of object $(n)$ and the number taken at any one time $(r)$. This helps us to understand which value goes where in the formula.**

**Now let's look at applying the formula to the example.**

For ${}_5C_3$:

${}_5C_3 = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!}$

Simplify.

${}_5C_3 = \frac{5(4)(3)(2)(1)}{(3 \cdot 2 \cdot 1)(2 \cdot 1)} = \frac{120}{12} = 10$

**There are 10 possible combinations.**

Example

Find ${}_6C_2$

***Step 1:*** Understand what ${}_6C_2$ means.

${}_6C_2 \Longleftarrow$ 6 items taken 2 at a time

***Step 2:*** Set up the problem.

${}_6C_2 = \frac{6!}{2!(6-2)!}$

***Step 3:*** Fill in the numbers and simplify.

${}_6C_2 = \frac{6(5)(4)(3)(2)(1)}{(2 \cdot 1)(4 \cdot 3 \cdot 2 \cdot 1)} = \frac{720}{48} = 15$

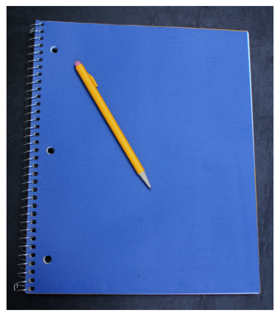
**There are 15 possible combinations.**

**12M. Lesson Exercises**

**Find the number of combinations in each example.**

1. ${}_5C_2$
2. ${}_4C_3$
3. ${}_6C_4$


***Check your answers with a friend.***



***Copy down the formula for figuring out combinations in your notebook***

## Real–Life Example Completed

***Decorating the Stage***

**Here is the original problem once again. Reread it and then figure out the decorations.**

The decorating committee is getting the stage ready for the Talent Show. There was a bunch of different decorating supplies ordered, and the students on the committee are working on figuring out the best way to decorate the stage.

They have four different colors of streamers to use to decorate.

Red

Blue

Green

Yellow

"I think four is too many colors. How about if we choose three of the four colors to decorate with?" Keith asks the group.

"I like that idea," Sara chimes in. "How many ways can we decorate the stage if we do that?"

The group begins to figure this out on a piece of paper.

**Combinations are arrangements where order does not make a difference. The decorating committee is selecting three colors from the possible four options. Therefore, the order of the colors doesn't matter.**

**We can use combination notation to figure out this problem.**

$_4C_3 = \frac{4!}{3!(4-3)!} = \frac{4(3)(2)(1)}{(3 \cdot 2 \cdot 1)(1)} = \frac{24}{6} = 4$

**There are four possible ways to decorate the stage.**

**Now that the students have this information, they can look at their color choices and vote on which combination they like best.**

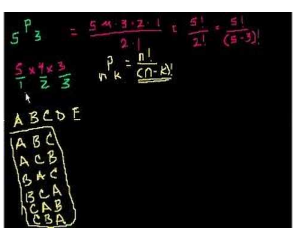
## Vocabulary

Here are the vocabulary words used in this lesson.

**Combination**
an arrangement of objects or events where order does not matter.

**Permutations**
an arrangement of objects or events where the order does matter.

## Technology Integration

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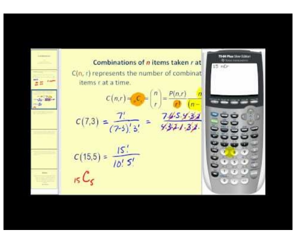
Khan Academy, Combinations



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James Sousa, Combinations

## Time to Practice

Directions: Write whether you are more likely to use permutations or combinations for each of the following examples.

1. A bag has 4 marbles: red, blue, yellow, and green. In how many different ways can you reach into the bag and draw out 2 marbles at once and drop them in a cup?

2. A bag contains 5 slips of paper with letters $A, B, C, D$, and $E$ written on them. Pull out one slip, mark down the letter and replace it in the bag. Do this 3 times so you have written 3 letters. How many different ways can you write the 3 letters?

3. Eight candidates are running for the 4-person Student Council. How many different Student Councils are possible?

4. Mario's gym locker uses the numbers 14, 6, and 32. How many different arrangements of the three numbers must Mario try to be sure he opens his locker?

5. Five horn players are running for 2 seats in a jazz band. How many different ways can the two horn players be chosen?

Directions: Use what you have learned about combinations to answer each question.

6. The Ace, King, Queen, and Jack of Spades are face down on a table. Draw three cards all at once. How many different 3-card hands can you draw?

7. How many different 4-player teams can you choose from a total of 5 volleyball players:

Andy, Randi, Sandy, Mandy, and Chuck?

8. How many different 3-player teams can you choose from a total of 5 volleyball players:

Andy, Randi, Sandy, Mandy, and Chuck?

9. A bag contains 6 slips of paper with letters $A, B, C, D, E$, and $F$ written on them. Pull out 4 slips. How many different 4-slip combinations can you get?

Directions: Evaluate each factorial.

10. $5!$

11. $4!$

12. $3!$

13. $8!$

14. $9!$

15. $6!$

Directions: Evaluate each combination using combination notation.

16. ${}_7C_2$

17. ${}_7C_6$

18. ${}_8C_4$

19. ${}_9C_6$

20. ${}_8C_3$

21. ${}_{10}C_7$

22. ${}_{12}C_9$

23. ${}_{11}C_9$

24. ${}_{16}C_{14}$

# 1.8 Distinguishable Permutations

**View this video on Distinguishable Permutations:**

**Distinguishable Permutations** http://www.educreations.com/lesson/view/distinguishable-permutations/10410554/?ref=appemail

**DISTINGUISHIBLE**

**PERMUTATIONS**

Ex) How many different permutations
can be made with the letters
A, A, and B ?

By the rule earlier, there are 3 ! = 6 different permutations possible, but some are repeats of others.

(Call the first A = $A_1$ and the second A = $A_2$)

$A_1A_2B$ $A_1BA_2$ $BA_1A_2$

$A_2A_1B$ $A_2BA_1$ $BA_2A_1$

But basically you have 3 different permutations there.

AAB ABA BAA

So there is only 3 distinguishable permutations.

**RULE**: The number of distinguishable permutations of *n* things where some things are repeated is:

$$\frac{N(\text{the total \# of letters})!}{(\text{the number of times each thing is repeated})!}$$

So for the last problem:

"How many different permutations can be made with the letters A, A, and B?"

n = 3 because there are 3 letters and

since there are two A's, divide by 2 !

Dist. Permutations = $\frac{3!}{2!} = 3$

Ex) Find the number of distinguishable permutations of the letters of the word ASPARAGUS.

Ans: Since there are 9 letters in the word, let n = 9. But since there are 3 A's and 2 S's, divide by 3 ! · 2!

Dist. Permutations = $\frac{9!}{3! \cdot 2!} = 30{,}240$

# CHAPTER 2 Sequences

## Chapter Outline



Discrete math is all about patterns, sequences, summing numbers, counting and probability. Many of these topics you will revisit in classes after Calculus. The goal here is to familiarize you with the important notation and the habits of thinking that accompany a mature way of looking at problems.

# 2.1 Introduction to Sequences

Here you will define patterns recursively and use recursion to solve problems.

**FACTORIAL NOTATION !**

**Def : If n is a positive integer (or 0), then n !**

**(read as "n factorial") is defined as**

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot n$$

- - - - - - - - - - - - - - - - - - - - - - - - - - -

**ex)** $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

$7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 5{,}040$

- - - - - - - - - - - - - - - - - - - - - - - - - - - -

**Note:**

$0! = 1$ **and** $1! = 1$

- - - - - - - - - - - - - - - - - - - - - - - - - - - - - -

Doing factorials on the Graphing Calculator

1. Enter the number you would like to take the factorial of.
2. Press the following keys to access the Math Probability menu

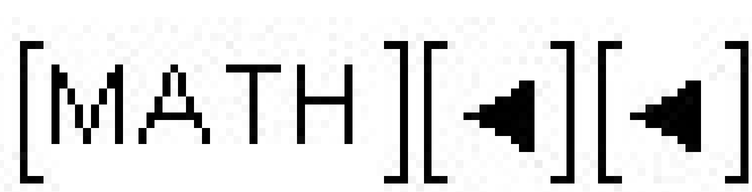



FIGURE 2.1

and press [4] to choose the factorial symbol (it looks like an exclamation point.)
There are more MATH submenus available on the TI-84 Plus C, if you use the TI-84 Plus, pay attention to the name of the submenu and use the left- and right-arrow keys to navigate to the correct one.

3. Press [ENTER] to evaluate the factorial.

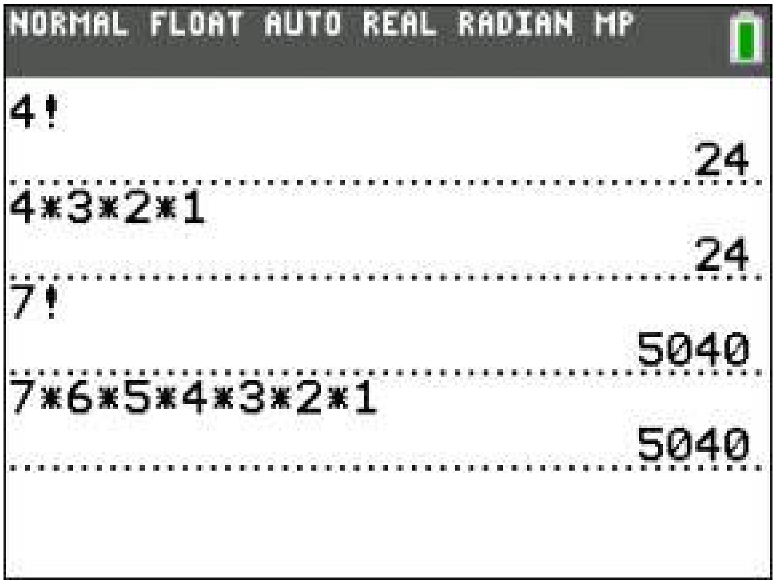



FIGURE 2.2

Some examples of what you are going to have to do with factorial:

**YOU TRY**

_

**SIMPLIFY**

**1)** $\frac{23!}{21!}$ **2)** $\frac{n!}{(n+1)!}$

**1)** $\frac{23!}{21!} = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot 21 \cdot 22 \cdot 23}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot 21} = \frac{22 \cdot 23}{1} = 506$

**2)** $\frac{n!}{(n+1)!}$

The S.A.T. approach is to substitute numbers in for n and try to guess the answer.

$\frac{n!}{(n+1)!}$

Write out what each stands for.

**2)** $\dfrac{\mathbf{n\,!}}{\mathbf{(n+1)\,!}} = \dfrac{\mathbf{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}}{\mathbf{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \cdot (n+1)}}$

$\dfrac{1}{n+1}$

Definition: A sequence is a pattern of numbers that is formed by plugging the positive integers (1,2,3,...) into a given rule.

Ex) $\mathbf{a_n = 2n - 1}$

To get the $1^{st}$ number (term) in the sequence, plug in n = 1.

$a_n = 2n - 1$

$a_1 = 2\,(1) - 1$

$a_1 = 2 - 1$

$a_1 = 1$

So the first number in the sequence is 1.

To get the $2^{nd}$ number (term) of the sequence, plug in n = 2.

$a_n = 2n - 1$

$a_2 = 2(2) - 1$

$a_2 = 4 - 1$

$a_2 = 3$

The second number in the sequence is 3.

To get the $3^{rd}$ number (term) of the sequence, plug in n = 3.

$a_n = 2n - 1$

$a_3 = 2(3) - 1$

$a_3 = 6 - 1$

$a_3 = 5$

So this sequence is

1,3,5,. . . , 2n-1, . . .

or the odd positive integers

**YOUR TURN**

Find the first 4 terms of

$$a_n = \frac{(-1)^n}{2n - 1}$$

Plug in 1 for every n to find the first term of the sequence,
plug in 2 for every n to find the second term of the sequence,
plug in 3 for every n to find the third term of the sequence,
plug in 4 for every n to find the fourth term of the sequence.

1 ) $a_n = \frac{(-1)^n}{2n - 1}$

$$\mathbf{a_1 = \frac{(-1)^1}{2(1) - 1} = \frac{-1}{2 - 1} = \frac{-1}{1} = -1}$$

$$\mathbf{a_2 = \frac{(-1)^2}{2(2) - 1} = \frac{1}{4 - 1} = \frac{1}{3} =}$$ **[U+2153]**

$$\mathbf{a_3 = \frac{(-1)^3}{2(3) - 1} = \frac{-1}{6 - 1} = \frac{-1}{5} = -1/5}$$

$$\mathbf{a_4 = \frac{(-1)^4}{2(4) - 1} = \frac{1}{8 - 1} = \frac{1}{7} = 1/7}$$

- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -

**Notes**:

1) $a_1$ stands for the first

number in the sequence,

$a_2$ stands for the second number,

$a_3$ stands for the $3^{rd}$ number, . . . ..

**$a_n$** stands for the given rule of the sequence.

($a_n$ is usually given to you)

2) Sequences are of infinite length. They continue forever unless you are told it is a finite sequence where you will be told at which term the sequence ends.

To have a graphing calculator find the terms of a sequence:

$$\mathbf{a_n = \frac{2^n}{n!}}$$

1) Put the calculator into sequence mode

MODE

Move cursor to SEQ and push ENTER

$2^{nd}$ QUIT –> exit's the mode

2) Push Y=

To enter the sequence

$$\mathbf{a_n = \frac{2^n}{n!}}$$

nMin = 1

(This tells the calculator to start plugging in n=1)

u(n) = This is where you type in the rule for the sequence

To enter in

$$\mathbf{a}_n = \frac{\mathbf{2}^n}{\mathbf{n\,!}}$$

Note: x,T,q,n is the variable button - most of the time x in an equation

3) Make sure

u($n$Min) = leave blank

Now to see the numbers in the sequence

$2^{nd}$ Graph (Table)

| n | u(n) |
|---|---|
| 1 | 2 |
| 2 | 2 |
| 3 | 1.3333 |
| 4 | .66667 |
| 5 | .26667 |

etc

View these two videos for more on sequences and having the graphing calculator find the terms of a sequence: **Introduction to Sequences** http://www.educreations.com/lesson/view/intro-to-sequences/10988807/?ref=appemail

**Having the calculator find the terms of a sequence** https://www.educreations.com/lesson/view/having-the-calculator-find-the-terms-of-a-sequence/11090207/

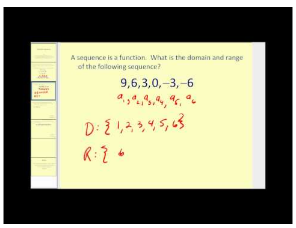
# 2.2 Arithmetic and Geometric Sequences

Here you will identify different types of sequences and use sequences to make predictions.

A sequence is a list of numbers with a common pattern. The common pattern in an arithmetic sequence is that the same number is added or subtracted to each number to produce the next number. The common pattern in a geometric sequence is that the same number is multiplied or divided to each number to produce the next number.

Are all sequences arithmetic or geometric?

### Watch This

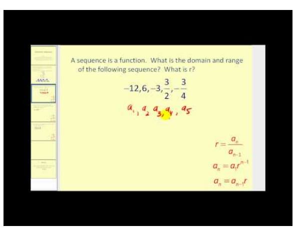



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/62261

http://www.youtube.com/watch?v=jExpsJTu9o8 James Sousa: Arithmetic Sequences



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/62263

http://www.youtube.com/watch?v=XHyeLKZYb2w James Sousa: Geometric Sequences

### Guidance

A sequence is just a list of numbers separated by commas. A sequence can be finite or infinite. If the sequence is infinite, the first few terms are followed by an ellipsis $(\ldots)$ indicating that the pattern continues forever.

**An infinite sequence**: $1, 2, 3, 4, 5, \ldots$

**A finite sequence:** $2, 4, 6, 8$

In general, you describe a sequence with subscripts that are used to index the terms. The $k^{th}$ term in the sequence is $a_k$.

$a_1, a_2, a_3, a_4, \ldots, a_k, \ldots$

Arithmetic sequences are defined by an initial value $a_1$ and a common difference $d$.

$$a_1 = a_1$$
$$a_2 = a_1 + d$$
$$a_3 = a_1 + 2d$$
$$a_4 = a_1 + 3d$$
$$\vdots$$
$$a_n = a_1 + (n-1)d$$

Geometric sequences are defined by an initial value $a_1$ and a common ratio $r$.

$$a_1 = a_1$$
$$a_2 = a_1 \cdot r$$
$$a_3 = a_1 \cdot r^2$$
$$a_4 = a_1 \cdot r^3$$
$$\vdots$$
$$a_n = a_1 \cdot r^{n-1}$$

If a sequence does not have a common ratio or a common difference, it is neither an arithmetic or a geometric sequence. You should still try to figure out the pattern and come up with a formula that describes it.

**Example A**

For each of the following three sequences, determine if it is arithmetic, geometric or neither.

a. $0.135, 0.189, 0.243, 0.297, \ldots$
b. $\frac{2}{9}, \frac{1}{6}, \frac{1}{8}, \ldots$
c. $0.54, 1.08, 3.24, \ldots$

**Solution:**

a. The sequence is arithmetic because the common difference is 0.054.
b. The sequence is geometric because the common ratio is $\frac{3}{4}$.
c. The sequence is not arithmetic because the differences between consecutive terms are 0.54 and 2.16 which are not common. The sequence is not geometric because the ratios between consecutive terms are 2 and 3 which are not common.

**Example B**

For the following sequence, determine the common ratio or difference, the next three terms, and the $2013^{th}$ term.

$\frac{2}{3}, \frac{5}{3}, \frac{8}{3}, \frac{11}{3}, \ldots$

**Solution:** The sequence is arithmetic because the difference is exactly 1 between consecutive terms. The next three terms are $\frac{14}{3}, \frac{17}{3}, \frac{20}{3}$. An equation for this sequence would is:

$a_n = \frac{2}{3} + (n-1) \cdot 1$

Therefore, the $2013^{th}$ term requires 2012 times the common difference added to the first term.

$a_{2013} = \frac{2}{3} + 2012 \cdot 1 = \frac{2}{3} + \frac{6036}{3} = \frac{6038}{3}$

**Example C**

For the following sequence, determine the common ratio or difference and the next three terms.

$\frac{2}{3}, \frac{4}{9}, \frac{6}{27}, \frac{8}{81}, \frac{10}{243}, \ldots$

**Solution:** This sequence is neither arithmetic nor geometric. The differences between the first few terms are $-\frac{2}{9}, -\frac{2}{9}, -\frac{10}{81}, -\frac{14}{243}$. While there was a common difference at first, this difference did not hold through the sequence. **Always check the sequence in multiple places to make sure that the common difference holds up throughout.**

The sequence is also not geometric because the ratios between the first few terms are $\frac{2}{3}, \frac{1}{2}, \frac{4}{9}$. These ratios are not common.

Even though you cannot get a common ratio or a common difference, it is still possible to produce the next three terms in the sequence by noticing the numerator is an arithmetic sequence with starting term of 2 and a common difference of 2. The denominators are a geometric sequence with an initial term of 3 and a common ratio of 3. The next three terms are:

$\frac{12}{3^6}, \frac{14}{3^7}, \frac{16}{3^8}$

**Concept Problem Revisited**

Example C shows that some patterns that use elements from both arithmetic and geometric series are neither arithmetic nor geometric. Two famous sequences that are neither arithmetic nor geometric are the Fibonacci sequence and the sequence of prime numbers.

**Fibonacci Sequence:** $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

**Prime Numbers:** $2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots$

## Vocabulary

A ***sequence*** is a list of numbers separated by commas.

The common pattern in an ***arithmetic sequence*** is that the same number is added or subtracted to each number to produce the next number. This is called the ***common difference.***

The common pattern in a ***geometric sequence*** is that the same number is multiplied or divided to each number to produce the next number. This is called the ***common ratio.***

## Guided Practice

1. What is the tenth term in the following sequence?

$-12, 6, -3, \frac{3}{2}, \ldots$

2. What is the tenth term in the following sequence?

$-1, \frac{2}{3}, \frac{7}{3}, 4, \frac{17}{3}, \ldots$

3. Find an equation that defines the $a_k$ term for the following sequence.

$0, 3, 8, 15, 24, 35, \ldots$

**Answers:**

1. The sequence is geometric and the common ratio is $-\frac{1}{2}$. The equation is $a_n = -12 \cdot \left(-\frac{1}{2}\right)^{n-1}$. The tenth term is:

$-12 \cdot \left(-\frac{1}{2}\right)^9 = \frac{3}{128}$

2. The pattern might not be immediately recognizable, but try ignoring the $\frac{1}{3}$ in each number to see the pattern a

different way.

$-3, 2, 7, 12, 17, \ldots$

You should see the common difference of 5. This means the common difference from the original sequence is $\frac{5}{3}$. The equation is $a_n = -1 + (n-1)\left(\frac{5}{3}\right)$. The $10^{th}$ term is:

$-1 + 9 \cdot \left(\frac{5}{3}\right) = -1 + 3 \cdot 5 = -1 + 15 = 14$

3. The sequence is not arithmetic nor geometric. It will help to find the pattern by examining the common differences and then the common differences of the common differences. This numerical process is connected to ideas in calculus.

0, 3, 8, 15, 24, 35

3, 5, 7, 9, 11

2, 2, 2, 2

Notice when you examine the common difference of the common differences the pattern becomes increasingly clear. Since it took *two* layers to find a constant function, this pattern is *quadratic* and very similar to the perfect squares.

$1, 4, 9, 16, 25, 36, \ldots$

The $a_k$ term can be described as $a_k = k^2 - 1$

### Practice

Use the sequence $1, 5, 9, 13, \ldots$ for questions 1-3.

1. Find the next three terms in the sequence.

2. Find an equation that defines the $a_k$ term of the sequence.

3. Find the $150^{th}$ term of the sequence.

Use the sequence $12, 4, \frac{4}{3}, \frac{4}{9}, \ldots$ for questions 4-6.

4. Find the next three terms in the sequence.

5. Find an equation that defines the $a_k$ term of the sequence.

6. Find the $17^{th}$ term of the sequence.

Use the sequence $10, -2, \frac{2}{5}, -\frac{2}{25}, \ldots$ for questions 7-9.

7. Find the next three terms in the sequence.

8. Find an equation that defines the $a_k$ term of the sequence.

9. Find the $12^{th}$ term of the sequence.

Use the sequence $\frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}, \ldots$ for questions 10-12.

10. Find the next three terms in the sequence.

11. Find an equation that defines the $a_k$ term of the sequence.

12. Find the $314^{th}$ term of the sequence.

13. Find an equation that defines the $a_k$ term for the sequence $4, 11, 30, 67, \ldots$

14. Explain the connections between arithmetic sequences and linear functions.

15. Explain the connections between geometric sequences and exponential functions.

# 2.3 Sigma Notation

Here you will learn how to represent the sum of sequences of numbers using sigma notation.

**SUMMATION**

**NOTATION**

∑

The above symbol (Sigma) is the symbol used in mathematics that tells you to **add** a finite number of terms.

**ex)** 6 < − − − − − − − − *upper limit of summation*

**∑ 3i**

**i = 1 <- - - - - - - - - - - lower limit of**

**summation**

This means to start at the lower limit of summation and plug into the "rule" and get a **number**. Then increase the lower limit of summation by one, plug into the rule and get another **number**. Keep increasing the lower limit of summation by one until you get to the upper limit of summation. Then add

all the **numbers**.

$$\sum_{i=1}^{6} 3i = 3(1) + 3(2) + 3(3) + 3(4) + 3(5) + 3(6)$$

3 + 6 + 9 + 12 + 15 + 18

= 63

**YOUR TURN**

ex)

$$\sum_{k=4}^{7} (1+k)^2$$

$(1+4)^2 + (1+5)^2 + (1+6)^2 + (1+7)^2$

$(5)^2 + (6)^2 + (7)^2 + (8)^2$

$25 + 36 + 49 + 64$

$174$

**DO NOW**

**1)** $\displaystyle\sum_{i=1}^{5} (2i^2 - 1)$

**3)** $\displaystyle\sum_{n=1}^{7} 6$

**2)** $\displaystyle\sum_{k=0}^{3} (-1)^k (k+1)$

**1)** $\displaystyle\sum_{i=1}^{5} (2i^2 - 1)$

**$2i^2 - 1$ $2i^2 - 1$ $2i^2 - 1$ $2i^2 - 1$ $2i^2 - 1$**

**$2(1)^2 - 1$ $2(2)^2 - 1$ $2(3)^2 - 1$ $2(4)^2 - 1$ $2(5)^2 - 1$**

**1 + 7 + 17 + 31 + 49**

**105**

**2)** $\sum_{k=0}^{3} (-1)^k (k+1)$

**$(-1)^k (k + 1)$ $(-1)^k (k + 1)$ $(-1)^k (k + 1)$ $(-1)^k (k + 1)$**

**$(-1)^0(0 + 1)$ $(-1)^1(1 + 1)$ $(-1)^2(2 + 1)$ $(-1)^3(3 + 1)$**

**(1)(1) (-1)(2) (1)(3) (-1)(4)**

**1 - 2 3 - 4**

- 2

**Note: The $(-1)^k$ part of the rule makes the terms alternate from positive to negative.**

**3)** $\sum_{n=1}^{7} 6$

**The "rule" is 6.**

**What do you get when you substitute n = 1**

**into the rule?**

**6**

**What do you get when you substitute n = 2**

**into the rule?**

**6**

**Each time you substitute into the rule**

**you get 6.**

**3)** $\sum_{n=0}^{7} 6$

**So since you are substituting 7 different times into the rule you are adding up 7 sixes.**

**6 + 6 + 6 + 6 + 6 + 6 + 6 = 42**

### Using the TI-84 Plus calculator's summation and logarithm templates

These templates can be found by pressing the up-arrow key to scroll in the MATH menu, or by pressing [ALPHA][WINDOW] to access the templates in the shortcut menu.

The summation template can be used to find the sum of a sequence. In math classrooms, this is commonly known as *Sigma notation.* The template should look exactly like a Sigma notation problem in your math textbook.

To use the summation template, insert


FIGURE 2.3

Notice the cursor has a blinking "A" indicating your calculator is in Alpha mode. Press the key that corresponds to the variable you want to use and press the right-arrow key. Enter the lower limit, press the right-arrow key, then enter the upper limit and press the right-arrow key again. Enter the expression and press [ENTER] to find the sum of the sequence as shown in the first line of the last screen.

# 2.4 Arithmetic Sequences

Here you will learn to compute finite arithmetic series more efficiently than just adding the terms together one at a time.

Def: An **arithmetic sequence** is a sequence whose consecutive terms have a COMMON DIFFERENCE.

Ex) 2,4,6,8,10,. . . ..

$a_2 - a_1 = 4 - 2 = 2$ <– common difference

$a_3 - a_2 = 6 - 4 = 2$ <– common difference

$a_4 - a_3 = 8 - 6 = 2$ <– common difference

View this video to learn more about Arithmetic Sequences:

**Arithmetic Sequences** http://www.educreations.com/lesson/view/arithmetic-sequences/11087373/?ref=appemail

1) 10,8,6,4,. . . ..

2) 1,2,4,8,16,. . . .

3) $a_n = 3^{n-1}$

4) $a_n = 5 + 3n$

**1) 10,8,6,4**

**$a_2 - a_1 = 8 - 10 = -2$**

**$a_3 - a_2 = 6 - 8 = -2$**

**$a_4 - a_3 = 4 - 6 = -2$**

**Yes, its arithmetic. d = - 2**

- - - - - - - - - - - - - - - - - - - - - - - - - - - - -

**2) 1,2,4,8,16,. . .**

**$a_2 - a_1 = 2 - 1 = 1$**

**$a_3 - a_2 = 4 - 2 = 2$**

**$a_4 - a_3 = 8 - 4 = 4$**

**Not an arithmetic sequence. There is NO common difference.**

- - - - - - - - - - - - - – - —- - - - - - - - - - - - - - - -

**3)** $\mathbf{a}_n = \mathbf{3}^{(n - 1)}$

**First, find the terms of the sequence on your calculator:**

$\mathbf{a}_n = \mathbf{3}^{(n - 1)}$

**1,3,9,27,. . .**

$a_2 - a_1 = 3 - 1 = 2$

$a_3 - a_2 = 9 - 3 = 6$

$a_4 - a_3 = 27 - 9 = 18$

There is no common difference, so it is NOT an arithmetic sequence.

**4)** $\mathbf{a}_n = \mathbf{5 + 3n}$

First, find the terms of the sequence.

8,11,14,17,. . .

$a_2 - a_1 = 11 - 8 = 3$

$a_3 - a_2 = 14 - 11 = 3$

$a_4 - a_3 = 17 - 14 = 3$

Yes it is an arithmetic sequence and d = 3.

## ARITHMETIC SEQUENCE FORMULA

**$a_n = a_1 \ \ d(n - 1)$**

**$a_1 = 1^{st}$ number in the sequence**

**d = common difference**

**n is always the subscript**

**The first step is ALWAYS to find d (if it is not given)**

**The second step is ALWAYS to find $a_1$ (if it is not given)**

**View this video to learn more about the Arithmetic Sequence Formula:**

**The Arithmetic Sequence Formula** http://www.educreations.com/lesson/view/the-arithmetic-sequence-formula/23252578/?ref=app

**Find the $406^{th}$ number in this pattern:**

**3, 9, 15, 21, 27, . . .**

**Is this an arithmetic sequence?**

**$a_2 - a_1 = 9 - 3 = 6$**

**$a_3 - a_2 = 15 - 9 = 6$**

**$a_4 - a_3 = 21 - 15 = 6$**

**Find the rule for the sequence:**

**3, 9, 15, 21, 27, . . .**

**$a_n = a_1 + d(n - 1)$**

**d = ? $a_1$ = ?**

**d = 6 $a_1$ = 3**

**Plug them into the formula:**

**$a_n = 3 + 6(n - 1)$**

**Now go back to the problem:**

**Find the 406$^{th}$ number in this pattern.**

**Use the calculator and get your answer from the table:**

**$a_{406}$ = 2433**

**lets PROGRAM the formula into your calculator**

**PRGM**

**NEW**

**CREATE NEW –> PUSH ENTER**

**NAME = ARITHMTC**

**: PUSH PRGM I/O 3 (DISP)**

**: DISP " AN = A1 + D(N - 1)"**

**2$^{nd}$ QUIT**

***Now when you want to see the formula***

**PRGM**

**1:ARITHMETIC –> Push ENTER**

**PrgmARITHMTC –> Push ENTER**

**Here is a video which instructs you how to store the Arithmetic Sequence Formula into the "Programs" section of your graphing calculator:**

**Programming the arithmetic sum formula into your calculator** https://www.educreations.com/lesson/view/programming-the-arithmetic-sum-formula-into-your-c/25139852/?s=4hpShU&ref=app

**Inserting Arithmetic Means**

**into a Sequence**

**ex) Insert 3 arithmetic means between 2 and 14.**

That means "put three numbers between 2 and 14 and form an arithmetic sequence"

**2 , ____ , ____ , ____ , 14**

**$a_1$ $a_2$ $a_3$ $a_4$ $a_5$**

First, find the common difference of the "sequence".

**Use the formula.**

**$a_n = a_1 + (n - 1)\,d$ Let n = 5**

**$a_5 = a_1 + (5 - 1)d$**

**$a_5 = a_1 + 4d$ Plug in $a_1 = 2$ and $a_5 = 14$**

**$14 = 2 + 4d$ Solve for d.**

$\underline{-2 \quad -2}$

$12 = 4d$

$d = 3$

2 , ____ , ____ , ____ , 14

$a_1$ $a_2$ $a_3$ $a_4$ $a_5$

Now add the "difference" to $a_1$.

+3 +3 +3 +3

2 , 5 , 8 , 11 , 14

**So the three arithmetic means are 5,8,11.**

ex) Find the sum of the integers from 1 to 100.

## The formula for the Sum of a Finite Arithmetic Sequence

$$\text{SUM} = \frac{n}{2}\left(2a_1 + d(n - 1)\right)$$

n = the number of "numbers" you are adding

$a_1$ = the first number in the sequence

d = the common difference in the sequence

Think of the numbers as a sequence:

1,2,3,. . . ,100

They form an ARITHMETIC sequence whose common difference is 1.

$$\text{SUM} = \frac{n}{2}(2a_1 + d(n-1))$$

n = 100 (the # of numbers you are adding)

$a_1 = 1$ (the first # in the sequence)

d = 1 (the common difference of the sequence)

$$\text{Sum} = \frac{100}{2}(2(1) + 1(100-1))$$

**Sum = 5,050**

View this video for more with the Arithmetic Sum Formula:

**Sum of an Arithmetic Sequence Formula** http://www.educreations.com/lesson/view/sum-of-an-arithmetic-sequence-formula/23252820/?ref=app

**PRGM**

**NEW**

**CREATE NEW –> PUSH ENTER**

**NAME = SUMARTHM**

**: PUSH PRGM I/O 3 (DISP)**

**: DISP " N/ 2(2A1+ D(N-1))"**

**2$^{nd}$ QUIT**

Practice

1. Sum the first 24 terms of the sequence $1,5,9,13,\dots$
2. Sum the first 102 terms of the sequence $7,9,11,13,\dots$
3. Sum the first 85 terms of the sequence $-3,-1,1,3,\dots$
4. Sum the first 97 terms of the sequence $\frac{1}{3},\frac{2}{3},1,\frac{4}{3},\dots$
5. Sum the first 56 terms of the sequence $-\frac{2}{3},\frac{1}{3},\frac{4}{3},\dots$
6. Sum the first 91 terms of the sequence $-8,-4,0,4,\dots$

Evaluate the following sums.

7. $\sum_{i=0}^{300} 3i+18$

8. $\sum_{i=0}^{215} 5i+1$

9. $\sum_{i=0}^{100} i-15$

10. $\sum_{i=0}^{85} -13i+1$

11. $\sum_{i=0}^{212} -2i+6$

12. $\sum_{i=0}^{54} 6i-9$

13. $\sum_{i=0}^{167} -5i+3$

14. $\sum_{i=0}^{341} 6i+102$

15. $\sum_{i=0}^{452} -7i-\frac{5}{2}$

View

# 2.5 Geometric Sequences

Here you will sum infinite and finite geometric series and categorize geometric series as convergent or divergent.

**Definition : A geometric sequence is a sequence whose consecutive terms have a COMMON RATIO (r).**

**Ex) 2,4,8,16,32,. . . .**

$\frac{a_2}{a_1} = \frac{4}{2} = 2 \quad \frac{a_3}{a_2} = \frac{8}{4} = 2 \quad \frac{a_4}{a_3} = \frac{16}{8} = 2$

**Another example of a geometric sequence:**

**4, 12, 36, 108,. . .**

**What is the common ratio ?**

**Note: To get the common ratio, you must divide the latter term by the former.**

$\frac{a_2}{a_1} = r$ **Not** $\frac{a_1}{a_2} \neq r$

$\frac{a_2}{a_1} = \frac{12}{4} = 3 \quad \frac{a_3}{a_2} = \frac{36}{12} = 3 \quad \frac{a_4}{a_3} = \frac{108}{36} = 3$

This is a geometric sequences with a common ratio $r = 3$.

**Ex) Determine whether the following sequences are geometric, arithmetic, or none of those. Find "r" or "d" accordingly:**

**1) 9 , -6 , 4 , -8/3 , . . . .**

**2) 1/3 , 2/3 , 3/3 , 4/3 ,. . . .**

**3)** $a_n = 4^{(n-1)}$

**4) $a_n = 4^n - 1$**

**1) 9 , -6 , 4 , -8/3 , . . . .**

$$\frac{a_2}{a_1} = \frac{-6}{9} = -2/3 \quad \frac{a_3}{a_2} = \frac{4}{-6} = -2/3$$

$$\frac{a_4}{a_3} = \frac{-8/3}{4} = -2/3$$

**Yes, it is a geometric sequence. r = -2/3**

**2) 1/3 , 2/3 , 3/3 , 4/3 ,. . . .**

$$\frac{a_2}{a_1} = \frac{2/3}{1/3} = 2$$

$$\frac{a_3}{a_2} = \frac{3/3}{2/3} = 3/2$$

$$\frac{a_4}{a_3} = \frac{4/3}{3/3} = 4/3$$

**This is not a geometric sequence.**

**it is an arithmetic sequence with d = 1/3)**

**$a_2 - a_1 = 2/3 - 1/3 = 1/3$**

**$a_3 - a_2 = 3/3 - 2/3 = 1/3$**

**$a_4 - a_3 = 4/3 - 3/3 = 1/3$**

**3) $a_n = 4^{(n-1)}$**

**First, you must find the first few terms of the sequence.**

**$a_n = 4^{(n-1)}$**

**Now, test and see if 1, 4, 16, 64, . . . is a geometric sequence.**

$$\frac{a_2}{a_1} = \frac{4}{1} = 4$$

$$\frac{a_3}{a_2} = \frac{16}{4} = 4$$

$$\frac{a_4}{a_3} = \frac{64}{16} = 4$$

**Yes, this is a geometric sequence. r = 4**

**4)** $a_n = 4^n - 1$

**Find the first few terms of the sequence.**

**Now see if 3, 15, 63, 255, . . . has a common ratio.**

$$\frac{a_2}{a_1} = \frac{15}{3} = 5$$

$$\frac{a_3}{a_2} = \frac{63}{15} = 4.2$$

$$\frac{a_4}{a_3} = \frac{255}{63} = 4.0$$

**This is NOT a geometric sequence b/c there is no common ratio.**

\- - - - - - - - - - - - - - - – - - - -– - - – - - – - – - -

**3, 15, 63, 255, . . .**

**<u>This is not an arithmetic either</u>**

**15 – 3 = 12**

**63 – 15 = 48**

**View this video to learn more about Geometric Sequences:**

**Geometric Sequences** http://www.educreations.com/lesson/view/geometric-sequences/11217703/?ref=appemail

**Find the first 5 terms of a geometric sequence if:**

**$a_1 = 3$ and r = 2.**

**3, ___ , ___ , ___ , ___ r = 2**

**$a_1$ $a_2$ $a_3$ $a_4$ $a_5$**

| | |
|---|---|
| **In a geometric sequence,** | **In an arithmetic seq.** |
| **–>means you multiply by "r"** | **–> means to add "d"** |
| **<– means you divide by "r"** | **<– means to subtract "d"** |

| | | | | |
|---|---|---|---|---|
| **3 x 2** | **6 x 2** | **12 x 2** | **24 x 2** | |
| **3 ,** | **6 ,** | **12 ,** | **24 ,** | **48** |
| **$a_1$** | **$a_2$** | **$a_3$** | **$a_4$** | **$a_5$** |

**So the first 5 terms are:**

**3 , 6 , 12 , 24 , 48**

**Geometric Sequence**

Formula

$a_n = a_1 \times r^{(n-1)}$

**To use this formula, all you have to know is $a_1$ and "r" and just plug them in. DON'T simplify.**

**ex) In a geometric sequence $a_1 = 5$ and r = 2. Find $a_6$ and $a_{10}$.**

$a_n = a_1 \cdot r^{(n-1)}$

*Plug in*: **$a_1 = 5$ and $r = 2$**

$$a_n = a_1 \cdot r^{(n ˘ 1)}$$
$$a_6 = 5 \cdot 2^{(6 ˘ 1)}$$
$$a_6 = 5 \cdot 2^{5}$$

**$a_6 = 160$.**

**So the $6^{th}$ term of the sequence is 160.**

**View this video for more help and practice with the Geometric Sequence Formula:**

**The Geometric Sequence Formula** http://www.educreations.com/lesson/view/the-geometric-sequence-formula/23253242/?ref=app

**ex) Find the sum of the first 6 terms of the sequence: 0.2, 0.06, 0.018, . . .**

## Sum of a Finite Geometric Sequence

**The sum of the first n terms of a geometric sequence can be found by using the following formula:**

$$\text{Sum} = a_1 \times \frac{1 - r^n}{1 - r}$$

**The "things" you need to know to use this formula is:**

**$a_1$ = the first # in the sequence**

**r = the ratio of the sequence**

**n = the # of terms you are adding**

**This is a geometric sequence with r =**

**0.2, 0.06, 0.018, . . .**

$$\frac{a_2}{} = \frac{0.06}{} = 0.3$$

$$\frac{a_1}{} \quad 0.2$$

$$\frac{a_3}{a_2} = \frac{0.018}{0.06} = 0.3$$

**So 0.2, 0.06, 0.018, . . . is a geometric sequence with r = 0.3.**

**ex) Find the sum of the first 6 terms of the sequence: 0.2, 0.06, 0.018, . . .**

$$\text{Sum} = a_1 \cdot \frac{1 - r^n}{1 - r}$$

**$a_1$ = 0.2 (the $1^{st}$ term)**

**r = 0.3 (the common ratio)**

**n = 6 (the number of numbers)**

$$\text{Sum} = 0.2 \cdot \frac{1 - 0.3^6}{1 - 0.3}$$

**Sum = 0.285506**

View this video for more help with the Sum of a Geometric Sequence Formula:

**Sum of a Geometric Sequence Formula** http://www.educreations.com/lesson/view/sum-of-a-geometric-sequence-formula/23253460/?ref=app

Practice

Find the sum of the first 15 terms for each geometric sequence below.

1. $5, 10, 20, \ldots$
2. $2, 8, 32, \ldots$
3. $5, \frac{5}{2}, \frac{5}{4}, \ldots$
4. $12, 4, \frac{4}{3}, \ldots$
5. $\frac{1}{3}, 1, 3, \ldots$

For each infinite geometric series, identify whether the series is convergent or divergent. If convergent, find the number where the sum converges.

6. $5+10+20+\cdots$

7. $2+8+32+\cdots$

8. $5+\frac{5}{2}+\frac{5}{4}+\cdots$

9. $12+4+\frac{4}{3}+\cdots$

10. $\frac{1}{3}+1+3+\cdots$

11. $6+2+\frac{2}{3}+\cdots$

12. You put \$5000 in a bank account at the end of every year for 30 years. The account earns 2% interest. How much do you have total at the end of 30 years?

13. You put \$300 in a bank account at the end of every year for 15 years. The account earns 4% interest. How much do you have total at the end of 10 years?

14. You put \$10,000 in a bank account at the end of every year for 12 years. The account earns 3.5% interest. How much do you have total at the end of 12 years?

15. Why don't infinite arithmetic series converge?

# 2.6 Sequence Word Problems

Here you will review counting using decision charts, permutations and combinations.

Word Problems

A movie theater has 30 rows of seats. There are 20 seats in the first row, 21 seats in the second row, 22 seats in the third row, etc. How many seats are in the entire movie theater ?

| 20 , | 21 , | 22 , | . . . . , | |
|---|---|---|---|---|
| row 1 | row 2 | row 3 | | row 30 |
| $a_1$ | $a_2$ | $a_3$ | | $a_{30}$ |

The number of seats in the rows forms an arithmetic sequence where the common difference is 1.

$$\textbf{Sum} = \frac{\mathbf{n}}{\mathbf{2}}\left(\mathbf{2a_1} + \mathbf{d(n-1)}\right)$$

n = 30 (the # of rows)

$a_1$ = 20 (the # of seats in the $1^{st}$ row)

d = 1 (the common difference)

Sum = 30/2 (2x20 + 1(30-1))

Sum = 1,035

For some more practice problems with sequence word problems, view this video:
**Sequence Word Problems Practice problems** http://www.educreations.com/lesson/view/page-270-38-41/12216958/?ref=appemail

SequencesYou learned that recursion, how most people intuitively see patterns, is where each term in a sequence is defined by the term that came before. You saw that terms in a pattern can also be represented as a function of their term number. You learned about two special types of patterns called arithmetic sequences and geometric sequences that have a wide variety of applications in the real world. You saw that series are when terms in a sequence are added together. A strong understanding of patterns helped you to count efficiently, which in turn allowed you to compute both basic and compound probabilities. Finally, you learned that induction is a method of proof that allows you to prove your own mathematical statements.

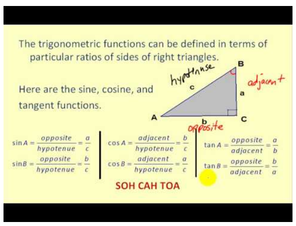
# CONCEPT 3 Right Triangle Trigonometry

Here you will learn the six right triangle ratios and how to use them to completely solve for the missing sides and angles of any right triangle.

Trigonometry is the study of triangles. If you know the angles of a triangle and one side length, you can use the properties of similar triangles and proportions to completely solve for the missing sides.

Imagine trying to measure the height of a flag pole. It would be very difficult to measure vertically because it could be several stories tall. Instead walk 10 feet away and notice that the flag pole makes a 65 degree angle with your feet. Using this information, what is the height of the flag pole?

### Watch This




MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/58116

http://www.youtube.com/watch?v=Ujyl_zQw2zE James Sousa: Introduction to Trigonometric Functions Using Triangles

### Guidance

The six trigonometric functions are sine, cosine, tangent, cotangent, secant and cosecant. *Opp* stands for the side opposite of the angle $\theta$, *hyp* stands for hypotenuse and *adj* stands for side adjacent to the angle $\theta$.

$\sin\theta = \frac{opp}{hyp}$

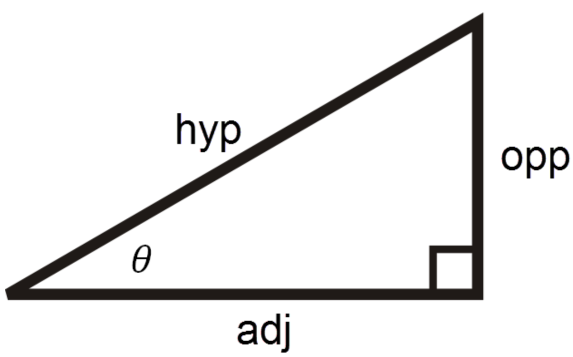
$\cos\theta = \frac{adj}{hyp}$

$\tan\theta = \frac{opp}{adj}$

$\cot\theta = \frac{adj}{opp}$

$\sec\theta = \frac{hyp}{adj}$

$\csc\theta = \frac{hyp}{opp}$



The reason why these trigonometric functions exist is because two triangles with the same interior angles will have side lengths that are always proportional. Trigonometric functions are used by identifying two known pieces of information on a triangle and one unknown, setting up and solving for the unknown. Calculators are important because the operations of sin, cos and tan are already programmed in. The other three (cot, sec and csc) are not usually in calculators because there is a reciprocal relationship between them and tan, cos and sec.

$\sin\theta = \frac{opp}{hyp} = \frac{1}{\csc\theta}$

$\cos\theta = \frac{adj}{hyp} = \frac{1}{\sec\theta}$

$\tan\theta = \frac{opp}{adj} = \frac{1}{\cot\theta}$

Keep in mind that your calculator can be in degree mode or radian mode. Be sure you can toggle back and forth so that you are always in the appropriate units for each problem.

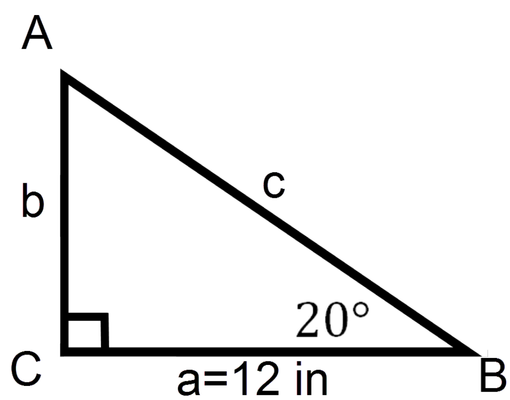
***Note: The images throughout this concept are not drawn to scale.***

## Here's a video to assist you with the basics of trigonometry:

**Basics of Trigonometry**

https://www.khanacademy.org/math/trigonometry/basic-trigonometry/basic_trig_ratios/v/basic-trigonometry

Given a right triangle with $a = 12\ in$, $m\angle B = 20^\circ$, and $m\angle C = 90^\circ$, find the length of the hypotenuse.




**Solution:**

$$\cos 20^\circ = \frac{12}{c}$$
$$c = \frac{12}{\cos 20^\circ} \approx 12.77\ in$$

**Here are some videos to assist you with evaluating trigonometry on the graphing calculator:**

**SOHCAHTOA using the Ti84 Plus**

https://www.youtube.com/watch?v=NLmS30e1YEo

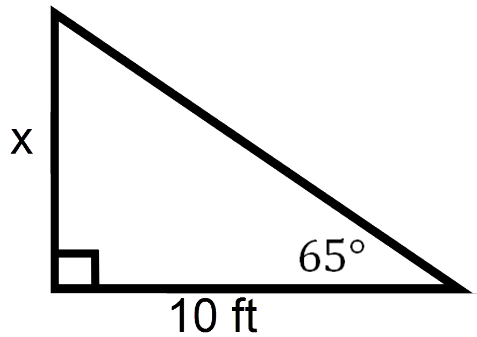
**Calculator Trig on the TI84 Plus**

https://www.youtube.com/watch?v=aS_0cizYRBw

**Concept Problem Revisited**

Instead walk 10 feet away and notice that the flag pole makes a $65^\circ$ angle with your feet.

If you walk 10 feet from the base of a flagpole and assume that the flagpole makes a 90° angle with the ground.




$$\tan 65^\circ = \frac{x}{10}$$
$$x = 10 \tan 65^\circ \approx 30.8\ ft$$

### Here is a video reviewing the reciprocal trig functions (secant, cosecant, cotangent)

**Reciprocal Trig Functions**

http://www.youtube.com/watch?v=25Yb2PUMDwk&oq=reciprocal%20identitie&gs_l=youtube..0.5j0j0i5.21282185.21287484.0.21290043.20.17.0.0.0.0.652.4745.2j6j2j3j1j3.17.0.eytns%2Cpt%3D-40%2Cn%3D2%2Cui%3Dt.1.0.0...1ac.1.11.youtube.Xny5N2IGwYA

### Vocabulary

The ***six trigonometric ratios*** are universal proportions that are always true of similar triangles (triangles with congruent corresponding angles).

θ ***(theta)*** is a Greek letter and is just a letter used in math to stand for an unknown angle.

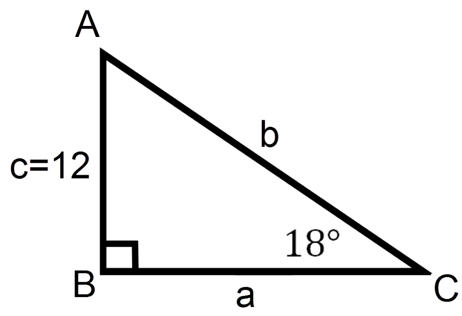
### Guided Practice

1. Given $\triangle ABC$ where $B$ is a right angle, $m\angle C = 18^\circ$, and $c = 12$. What is $a$?

2. Given $\triangle MNO$ where $O$ is a right angle, $m = 12$, and $n = 14$. What is the measure of angle $M$?

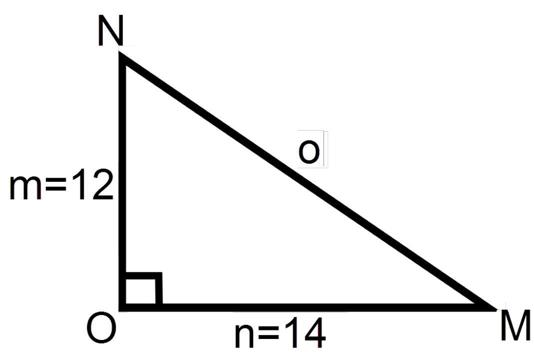
**Answers:**

1. Drawing out this triangle, it looks like:




$$\tan 18^\circ = \frac{12}{a}$$
$$a = \frac{12}{\tan 18^\circ} \approx 36.9$$

2. Drawing out the triangle, it looks like:




$$\tan M = \frac{12}{14}$$
$$M = \tan^{-1}\left(\frac{12}{14}\right) \approx 0.7\ radian \approx 40.6^\circ$$

### Practice

For 1-15, information about the sides and/or angles of right triangle *ABC* is given. Completely solve the triangle (find all missing sides and angles) to 1 decimal place.

**TABLE 3.1:**

| Problem Number | $A$ | $B$ | $C$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|---|---|
| 1. | 90° | | | | 4 | 7 |
| 2. | 90° | | 37° | 18 | | |
| 3. | | 90° | 15° | | 32 | |
| 4. | | | 90° | 6 | | 11 |
| 5. | 90° | 12° | | 19 | | |

TABLE 3.1: (continued)

| | | | | | | |
|---|---|---|---|---|---|---|
| 6. | | $90^\circ$ | | | 17 | 10 |
| 7. | $90^\circ$ | $10^\circ$ | | | 2 | |
| 8. | $4^\circ$ | $90^\circ$ | | 0.3 | | |
| 9. | $\frac{\pi}{2}$ radian | | 1 radian | | | 15 |
| 10. | | $\frac{\pi}{2}$ radian | | 12 | 15 | |
| 11. | | | $\frac{\pi}{2}$ radian | | 9 | 14 |
| 12. | $\frac{\pi}{4}$ radian | $\frac{\pi}{4}$ radian | | | 5 | |
| 13. | $\frac{\pi}{2}$ radian | | | 26 | 13 | |
| 14. | | $\frac{\pi}{2}$ radian | | | 19 | 16 |
| 15. | | | $\frac{\pi}{2}$ radian | 10 | | $10\sqrt{2}$ |

## References


1. CK-12 Foundation. . CCSA
2. CK-12 Foundation. . CCSA
3. CK-12 Foundation. . CCSA
4. CK-12 Foundation. . CCSA

# CONCEPT 4 Applications of Basic Triangle Trigonometry

Here you will apply your knowledge of trigonometry and problem solving in context.

First view the video reviewing the basics of how to find sides and angles of a right triangle using SOHCAHTOA.

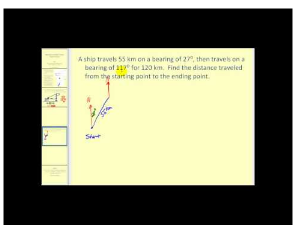
**Finding Sides and Angles of a Triangle using SOHCAHTOA**

http://www.educreations.com/lesson/view/finding-sides-angles-of-triangles/12804373/?ref=appemail

Deciding when to use SOH, CAH, TOA, Law of Cosines or the Law of Sines is not always obvious. Sometimes more than one approach will work and sometimes correct computations can still lead to incorrect results. This is because correct interpretation is still essential.

If you use both the Law of Cosines and the Law of Sines on a triangle with sides 4, 7, 10 you end up with conflicting answers. Why?

## Watch This



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/58154

http://www.youtube.com/watch?v=-QOEcnuGQwo James Sousa: Solving Right Triangles-Part 2 Applications

## Guidance

When applying trigonometry, it is important to have a clear toolbox of mathematical techniques to use. Some of the techniques may be review like the fact that all three angles in a triangle sum to be $180^\circ$, other techniques may be new like the Law of Cosines. There also may be some properties that are true and make sense but have never been formally taught.

**Toolbox:**

- The three angles in a triangle sum to be $180^\circ$.
- There are $360^\circ$ in a circle and this can help us interpret negative angles as positive angles.
- The Pythagorean Theorem states that for legs $a, b$ and hypotenuse $c$ in a right triangle, $a^2+b^2=c^2$.
- The Triangle Inequality Theorem states that for any triangle, the sum of any two of the sides must be greater than the third side.
- The Law of Cosines: $c^2=a^2+b^2-2ab\cos C$
- The Law of Sines: $\frac{a}{\sin A}=\frac{b}{\sin B}$ or $\frac{\sin A}{a}=\frac{\sin B}{b}$ (Be careful for the ambiguous case)
- SOH CAH TOA is a mnemonic device to help you remember the three original trig functions:

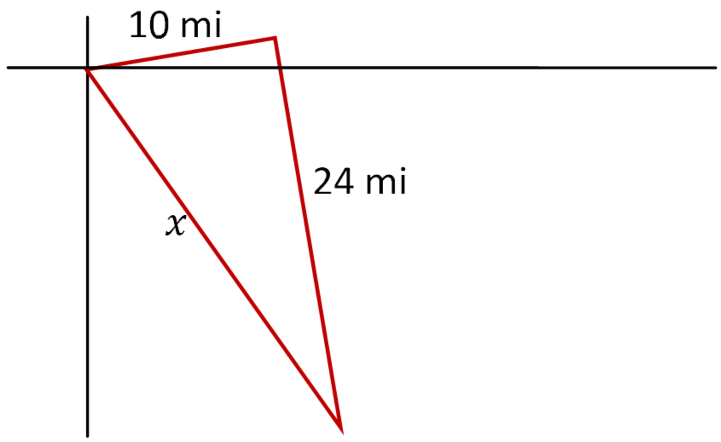
$$\sin\theta=\frac{opp}{hyp} \quad \cos\theta=\frac{adj}{hyp} \quad \tan\theta=\frac{opp}{adj}$$

- 30-60-90 right triangles have side ratios $x, x\sqrt{3}, 2x$
- 45-45-90 right triangles have side ratios $x, x, x\sqrt{2}$
- Pythagorean number triples are exceedingly common and should always be recognized in right triangle problems. Examples of triples are 3, 4, 5 and 5, 12, 13.

**Example A**

Bearing is how direction is measured at sea. North is $0^\circ$, East is $90^\circ$, South is $180^\circ$ and West is $270^\circ$. A ship travels 10 miles at a bearing of $88^\circ$ and then turns $90^\circ$ to the right to avoid an iceberg for 24 miles. How far is the ship from its original position?

**Solution:** First draw a clear sketch.



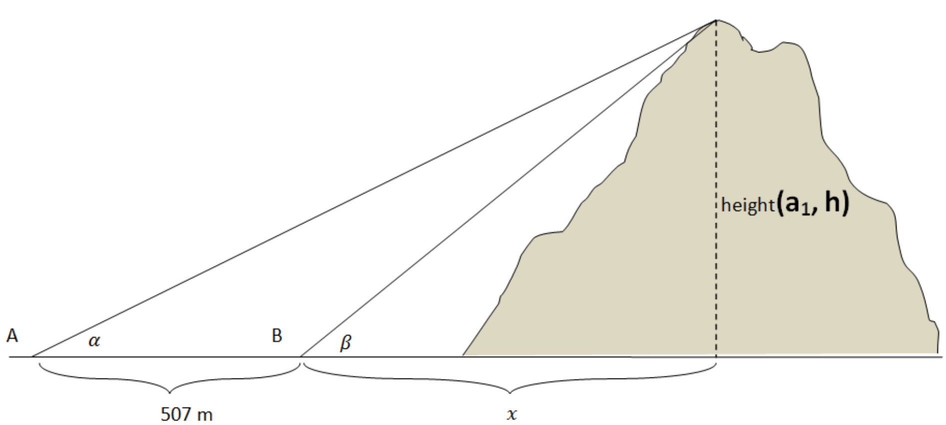

Next, recognize the right triangle with legs 10 and 24. This is a multiple of the 5, 12, 13 Pythagorean number triple and so the distance $x$ must be 26 miles.

**Example B**

A surveying crew is given the job of verifying the height of a cliff. From point $A$, they measure an angle of elevation to the top of the cliff to be $\alpha = 21.567^\circ$. They move 507 meters closer to the cliff and find that the angle to the top of the cliff is now $\beta = 25.683^\circ$. How tall is the cliff?

*Note that* $\alpha$ *is just the Greek letter alpha and in this case it stands for the number* $21.567^\circ$. $\beta$ *is the Greek letter beta and it stands for the number* $25.683^\circ$.

**Solution:** First, sketch the image and label what you know.



Next, because the height is measured at a right angle with the ground, set up two equations. Remember that $\alpha$ and $\beta$ are just numbers, not variables.

$$\tan\alpha = \frac{h}{507+x}$$
$$\tan\beta = \frac{h}{x}$$

Both of these equations can be solved for $h$ and then set equal to each other to find $x$.

$$\begin{aligned}
h = \tan\alpha(507+x) &= x\tan\beta \\
507\tan\alpha + x\tan\alpha &= x\tan\beta \\
507\tan\alpha &= x\tan\beta - x\tan\alpha \\
507\tan\alpha &= x(\tan\beta - \tan\alpha) \\
x = \frac{507\tan\alpha}{\tan\beta - \tan\alpha} &= \frac{507\tan 21.567^\circ}{\tan 25.683^\circ - \tan 21.567^\circ} \approx 228.7 \text{ } meters
\end{aligned}$$

Since the problem asked for the height, you need to substitute $x$ back and solve for $h$.

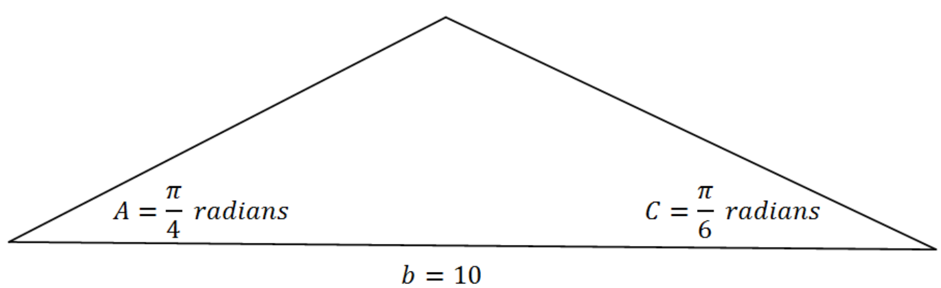
$h = x\tan\beta = 228.7\tan 25.683^\circ \approx 109.99 \text{ } meters$

**Example C**

Given a triangle with SSS or SAS you know to use the Law of Cosines. In triangles where there are corresponding angles and sides like AAS or SSA it makes sense to use the Law of Sines. What about ASA?

Given $\Delta ABC$ with $A = \frac{\pi}{4} \text{ } radians, C = \frac{\pi}{6} \text{ } radians$ and $b = 10 \text{ } in$ what is $a$?

**Solution:** First, draw a picture.




The sum of the angles in a triangle is $180^\circ$. Since this problem is in radians you either need to convert this rule to radians, or convert the picture to degrees.

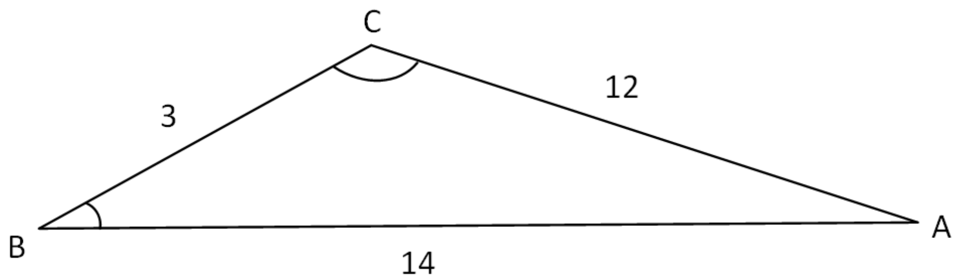
$$A = \frac{\pi}{4} \cdot \frac{180^\circ}{\pi} = 45^\circ$$
$$C = \frac{\pi}{6} \cdot \frac{180^\circ}{\pi} = 30^\circ$$

The missing angle must be $\angle B = 105^\circ$. Now you can use the Law of Sines to solve for $a$.

$$\frac{\sin 105^\circ}{10} = \frac{\sin 45^\circ}{a}$$
$$a = \frac{10\sin 45^\circ}{\sin 105^\circ} \approx 7.32\ in$$

**Concept Problem Revisited**

Sometimes when using the Law of Sines you can get answers that do not match the Law of Cosines. Both answers can be correct computationally, but the Law of Sines may involve interpretation when the triangle is obtuse. The Law of Cosines does not require this interpretation step.




First, use Law of Cosines to find $\angle B$:

$$12^2 = 3^2 + 14^2 - 2\cdot 3\cdot 14\cdot \cos B$$
$$\angle B = \cos^{-1}\left(\frac{12^2 - 3^2 - 14^2}{-2\cdot 3\cdot 14}\right) \approx 43.43\ldots^\circ$$

Then, use Law of Sines to find $\angle C$. Use the unrounded value for $B$ even though a rounded value is shown.

$$\frac{\sin 43.43^\circ}{12} = \frac{\sin C}{14}$$
$$\frac{14\sin 43.43^\circ}{12} = \sin C$$
$$\angle C = \sin^{-1}\left(\frac{14\sin 43.43^\circ}{12}\right) \approx 53.3^\circ$$

Use the Law of Cosines to double check $\angle C$.

$$14^2 = 3^2 + 12^2 - 2\cdot 3\cdot 12\cdot \cos C$$
$$C = \cos^{-1}\left(\frac{14^2 - 3^2 - 12^2}{-2\cdot 3\cdot 12}\right) \approx 126.7^\circ$$

Notice that the last two answers do not match, but they are supplementary. This is because this triangle is obtuse and the $\sin^{-1}\left(\frac{opp}{hyp}\right)$ function is restricted to only producing acute angles.

### Vocabulary

***Angle of elevation*** is the angle at which you view an object above the horizon.

***Angle of depression*** is the angle at which you view an object below the horizon. This can be thought of negative angles of elevation.

***Bearing*** is how direction is measured at sea. North is $0^\circ$, East is $90^\circ$, South is $180^\circ$ and West is $270^\circ$.

Greek letters ***alpha*** and ***beta*** $(\alpha, \beta)$ are often used as placeholders for known angles. Unknown angles are often referred to as $\theta$ ***(theta).***

***ASA*** refers to the situation from geometry when there are two known angles in a triangle and one known side that is between the known angles.

### Guided Practice

1. The angle of depression of a boat in the distance from the top of a lighthouse is $\frac{\pi}{10}$. The lighthouse is 200 feet tall. Find the distance from the base of the lighthouse to the boat.

2. From the third story of a building (50 feet) David observes a car moving towards the building driving on the streets below. If the angle of depression of the car changes from $21^\circ$ to $45^\circ$ while he watches, how far did the car travel?

3. If a boat travels 4 miles SW and then 2 miles NNW, how far away is it from its starting point?

**Answers:**

1. When you draw a picture, you see that the given angle $\frac{\pi}{10}$ is not directly inside the triangle between the lighthouse, the boat and the base of the lighthouse. It is complementary to the angle you need.

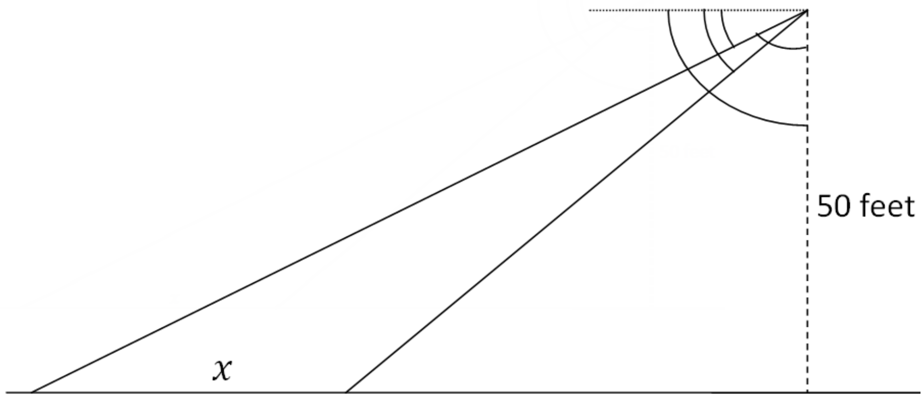
$$\begin{aligned}\frac{\pi}{10} + \theta &= \frac{\pi}{2}\\ \theta &= \frac{2\pi}{5}\end{aligned}$$

Now that you have the angle, use tangent to solve for $x$.

$$\begin{aligned}\tan\frac{2\pi}{5} &= \frac{x}{200}\\ x &= 200\tan\frac{2\pi}{5} \approx 615.5\ldots ft\end{aligned}$$

Alternatively, you could have noticed that $\frac{\pi}{10}$ is alternate interior angles with the angle of elevation of the lighthouse from the boat's perspective. This would yield the same distance for $x$.

2. Draw a very careful picture:



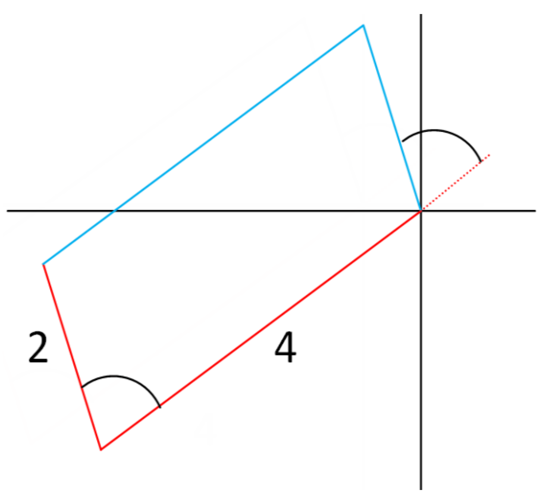

In the upper right corner of the picture there are four important angles that are marked with angles. The measures of these angles from the outside in are $90^\circ, 45^\circ, 21^\circ, 69^\circ$. There is a 45-45-90 right triangle on the right, so the base must also be 50. Therefore you can set up and solve an equation for $x$.

$$\tan 69^\circ = \frac{x+50}{50}$$
$$x = 50\tan 69^\circ - 50 \approx 80.25\ldots ft$$

The hardest part of this problem is drawing a picture and working with the angles.

3. 4 miles SW and then 2 miles NNW

Translate SW and NNW into degrees bearing. SW is a bearing of $225^\circ$ and NNW is a bearing of $315^\circ$. Draw a picture in two steps. Draw the original 4 miles traveled and draw the second 2 miles traveled from the origin. Then translate the second leg of the trip so it follows the first leg. This way you end up with a parallelogram, which has interior angles that are easier to calculate.




The angle between the two red line segments is $67.5^\circ$ which can be seen if the red line is extended past the origin.

The shorter diagonal of the parallelogram is the required unknown information.

$$x^2 = 4^2 + 2^2 - 2 \cdot 4 \cdot 2 \cdot \cos 67.5^\circ$$
$$x \approx 3.7 \; miles$$

### Practice

The angle of depression of a boat in the distance from the top of a lighthouse is $\frac{\pi}{6}$. The lighthouse is 150 feet tall. You want to find the distance from the base of the lighthouse to the boat.

1. Draw a picture of this situation.

2. What methods or techniques will you use?

3. Solve the problem.

From the third story of a building (60 feet) Jeff observes a car moving towards the building driving on the streets below. The angle of depression of the car changes from $34^\circ$ to $62^\circ$ while he watches. You want to know how far the car traveled.

4. Draw a picture of this situation.

5. What methods or techniques will you use?

6. Solve the problem.

A boat travels 6 miles NW and then 2 miles SW. You want to know how far away the boat is from its starting point.

7. Draw a picture of this situation.

8. What methods or techniques will you use?

9. Solve the problem.

You want to figure out the height of a building. From point $A$, you measure an angle of elevation to the top of the building to be $\alpha = 10^\circ$. You move 50 feet closer to the building to point $B$ and find that the angle to the top of the building is now $\beta = 60^\circ$.

10. Draw a picture of this situation.

11. What methods or techniques will you use?

12. Solve the problem.

13. Given $\Delta ABC$ with $A = 40^\circ, C = 65^\circ$ and $b = 8\ in$, what is $a$?

14. Given $\Delta ABC$ with $A = \frac{\pi}{3}\ radians, C = \frac{\pi}{8}\ radians$ and $b = 12\ in$ what is $a$?

15. Given $\Delta ABC$ with $A = \frac{\pi}{6}\ radians, C = \frac{\pi}{4}\ radians$ and $b = 20\ in$ what is $a$?

## References

1. CK-12 Foundation. . CCSA
2. CK-12 Foundation. . CCSA
3. CK-12 Foundation. . CCSA
4. CK-12 Foundation. . CCSA
5. CK-12 Foundation. . CCSA
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# CONCEPT 5 Law of Cosines

Here you will solve non-right triangles with the Law of Cosines.

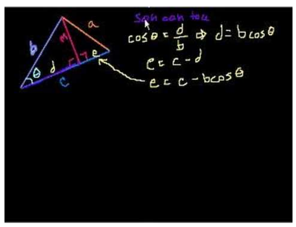
Here is a video explaining the Law of Cosines:

**LAW OF COSINES**

http://www.youtube.com/watch?v=pxVAWYOe34c

The Law of Cosines is a generalized Pythagorean Theorem that allows you to solve for the missing sides and angles of a triangle even if it is not a right triangle. Suppose you have a triangle with sides 11, 12 and 13. What is the measure of the angle opposite the 11?

### Watch This



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/58142

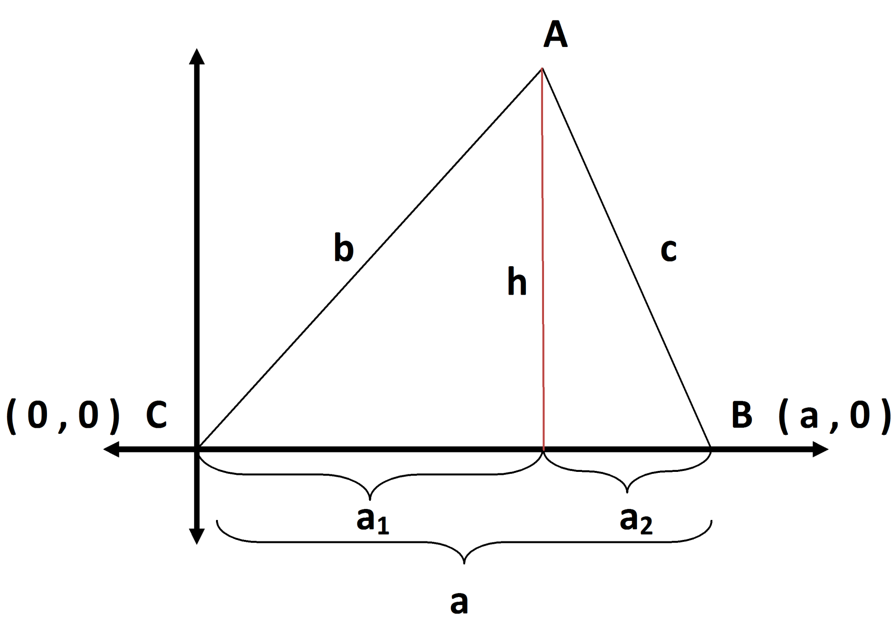
http://www.youtube.com/watch?v=pGaDcOMdw48 Khan Academy: Law of Cosines

### Guidance

The Law of Cosines is:

$c^2 = a^2 + b^2 - 2ab \cdot \cos C$

It is important to understand the proof:



You know four facts from the picture:

$a = a_1 + a_2 \qquad (1)$

$b^2 = a_1^2 + h^2 \qquad (2)$

$c^2 = a_2^2 + h^2 \qquad (3)$

$\cos C = \frac{a_1}{b} \qquad (4)$

Once you verify for yourself that you agree with each of these facts, check algebraically that these next two facts must be true.

$a_2 = a - a_1 \qquad (5,\ \text{from } 1)$

$a_1 = b \cdot \cos C \qquad (6,\ \text{from } 4)$

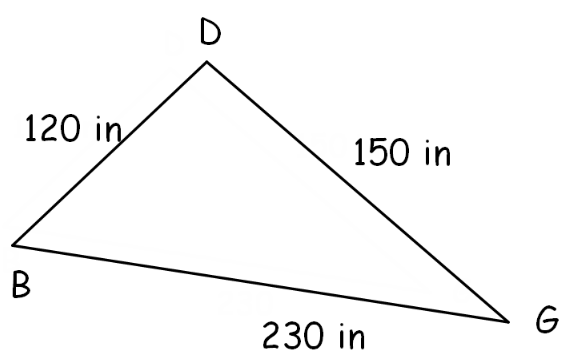
Now the Law of Cosines is ready to be proved using substitution, FOIL, more substitution and rewriting to get the order of terms right.

$$
\begin{array}{ll}
c^2 = a_2^2 + h^2 & \text{(3 again)} \\
c^2 = (a - a_1)^2 + h^2 & \text{(substitute using 5)} \\
c^2 = a^2 - 2a \cdot a_1 + a_1^2 + h^2 & \text{(FOIL)} \\
c^2 = a^2 - 2a \cdot b \cdot \cos C + a_1^2 + h^2 & \text{(substitute using 6)} \\
c^2 = a^2 - 2a \cdot b \cdot \cos C + b^2 & \text{(substitute using 2)} \\
c^2 = a^2 + b^2 - 2ab \cdot \cos C & \text{(rearrange terms)}
\end{array}
$$

There are only two types of problems in which it is appropriate to use the Law of Cosines. The first is when you are given all three sides of a triangle and asked to find an unknown angle. This is called SSS like in geometry. The second situation where you will use the Law of Cosines is when you are given two sides and the included angle and you need to find the third side. This is called SAS.

**Example A**

Determine the measure of angle $D$.

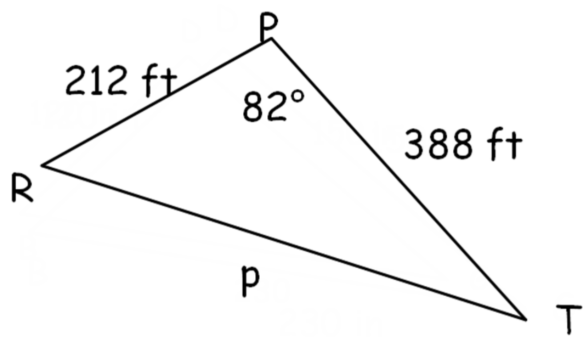



**Solution:** It is necessary to set up the Law of Cosines equation very carefully with $D$ corresponding to the opposite side of 230. The letters are not $ABC$ like in the proof, but those letters can always be changed to match the problem as long as the angle in the cosine corresponds to the side used in the left side of the equation.

$$
\begin{aligned}
c^2 &= a^2 + b^2 - 2ab \cdot \cos C \\
230^2 &= 120^2 + 150^2 - 2 \cdot 120 \cdot 150 \cdot \cos D \\
230^2 - 120^2 - 150^2 &= -2 \cdot 120 \cdot 150 \cdot \cos D \\
\frac{230^2 - 120^2 - 150^2}{-2 \cdot 120 \cdot 150} &= \cos D \\
D &= \cos^{-1}\left(\frac{230^2 - 120^2 - 150^2}{-2 \cdot 120 \cdot 150}\right) \approx 116.4^\circ \approx 2.03 \textit{ radians}
\end{aligned}
$$

**Example B**

Determine the length of side $p$.

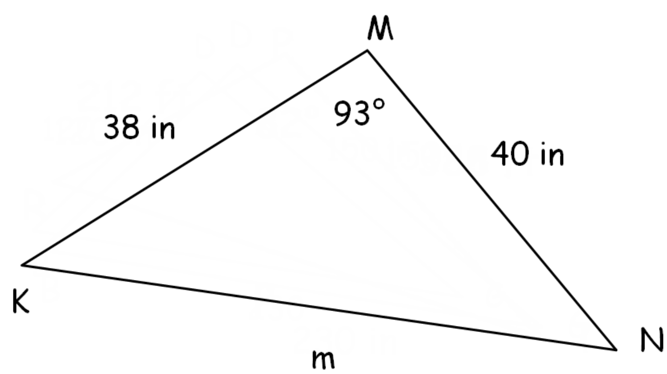



**Solution:**

$$
\begin{aligned}
c^2 &= a^2 + b^2 - 2ab \cdot \cos C \\
p^2 &= 212^2 + 388^2 - 2 \cdot 212 \cdot 388 \cdot \cos 82^\circ \\
p^2 &\approx 194192.02\ldots \\
p &\approx 440.7
\end{aligned}
$$

**Example C**

Determine the degree measure of angle $N$.



**Solution:** This problem must be done in two parts. First apply the Law of Cosines to determine the length of side $m$. This is a SAS situation like Example B. Once you have all three sides you will be in the SSS situation like in Example A and can apply the Law of Cosines again to find the unknown angle $N$.

$$\begin{aligned} c^2 &= a^2+b^2-2ab\cdot\cos C \\ m^2 &= 38^2+40^2-2\cdot 38\cdot 40\cdot\cos 93^\circ \\ m^2 &\approx 3203.1\ldots \\ m &\approx 56.59\ldots \end{aligned}$$

Now that you have all three sides you can apply the Law of Cosines again to find the unknown angle $N$. Remember to match angle $N$ with the corresponding side length of 38 inches. It is also best to store $m$ into your calculator and use the unrounded number in your future calculations.

$$\begin{aligned} c^2 &= a^2+b^2-2ab\cdot\cos C \\ 38^2 &= 40^2+(56.59\ldots)^2-2\cdot 40\cdot(56.59\ldots)\cdot\cos N \\ 38^2-40^2-(56.59\ldots)^2 &= -2\cdot 40\cdot(56.59\ldots)\cdot\cos N \\ \frac{38^2-40^2-(56.59\ldots)^2}{-2\cdot 40\cdot(56.59\ldots)} &= \cos N \\ N &= \cos^{-1}\left(\frac{38^2-40^2-(56.59\ldots)^2}{-2\cdot 40\cdot(56.59\ldots)}\right)\approx 42.1^\circ \end{aligned}$$

**Concept Problem Revisited**

A triangle that has sides 11, 12 and 13 is not going to be a right triangle. In order to solve for the missing angle you need to use the Law of Cosines because this is a SSS situation.

$$\begin{aligned} c^2 &= a^2+b^2-2ab\cdot\cos C \\ 11^2 &= 12^2+(13)^2-2\cdot 12\cdot 13\cdot\cos C \\ C &= \cos^{-1}\left(\frac{11^2-12^2-13^2}{-2\cdot 12\cdot 13}\right)\approx 52.02\ldots^\circ \end{aligned}$$

## Vocabulary

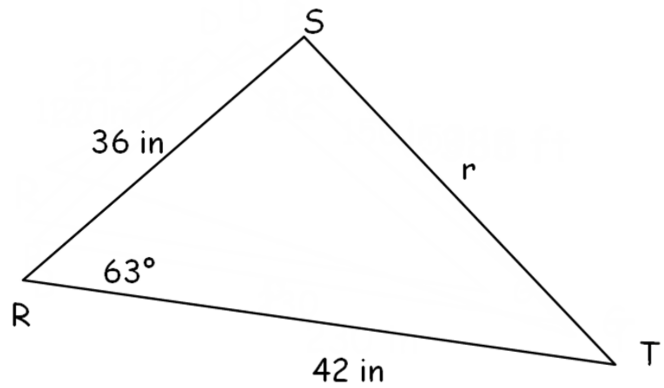
The **Law of Cosines** is a generalized Pythagorean Theorem that allows you to solve for the missing sides and angles of a triangle even if it is not a right triangle.

**SSS** refers to Side, Side, Side and refers to a property of congruent triangles in geometry. In this case it refers to the fact that all three sides are known in the problem.

**SAS** refers to Side, Angle, Side and refers to a property of congruent triangles in geometry. In this case it refers to the fact that the known quantities of a triangle are two sides and the included angle.

**Included angle** is the angle between two sides.

### Guided Practice




1. Determine the length of side $r$.

2. Determine the measure of angle $T$ in degrees.

3. Determine the measure of angle $S$ in radians.

**Answers:**

1. $r^2 = 36^2 + 42^2 - 2 \cdot 36 \cdot 42 \cdot \cos 63$

$r = 41.07\ldots$

2. $36^2 = (41.07\ldots)^2 + 42^2 - 2 \cdot (41.07\ldots) \cdot 42 \cdot \cos T$

$T \approx 51.34\ldots^\circ$

3. You could repeat the process from the previous question, or use the knowledge that the three angles in a triangle add up to 180.

$$63 + 51.34\ldots + S = 180$$
$$S \approx 65.65^\circ \cdot \frac{\pi}{180^\circ} \approx 1.145\ldots radians$$

### Practice

For all problems, find angles in degrees rounded to one decimal place.

In $\Delta ABC, a = 12, b = 15$, and $c = 20$.

1. Find the measure of angle $A$.

2. Find the measure of angle $B$.

3. Find the measure of angle $C$.

4. Find the measure of angle $C$ in a different way.

In $\Delta DEF, d = 20, e = 10$, and $f = 16$.

5. Find the measure of angle $D$.

6. Find the measure of angle $E$.

7. Find the measure of angle $F$.

In $\Delta GHI, g = 19, \angle H = 55^\circ$, and $i = 12$.

8. Find the length of $h$.

9. Find the measure of angle $G$.

10. Find the measure of angle $I$.

11. Explain why the Law of Cosines is connected to the Pythagorean Theorem.

12. What are the two types of problems where you might use the Law of Cosines?

Determine whether or not each triangle is possible.

13. $a = 5, b = 6, c = 15$

14. $a = 1, b = 5, c = 4$

15. $a = 5, b = 6, c = 10$

## References

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# CONCEPT 6 Law of Sines

Here you will further explore solving non-right triangles in cases where a corresponding side and angle are given using the Law of Sines.

Here are two videos reviewing the Law of Sines:

**LAW OF SINES**

**Video 1**

http://www.youtube.com/watch?v=bDPRWJdVzfs

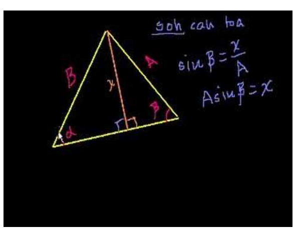
**Video 2**

http://www.youtube.com/watch?v=yVquId7xJQY

When given a right triangle, you can use basic trigonometry to solve for missing information. When given SSS or SAS, you can use the Law of Cosines to solve for the missing information. But what happens when you are given two sides of a triangle and an angle that is not included? There are many ways to show two triangles are congruent, but SSA is not one of them. Why not?

### Watch This

http://www.youtube.com/watch?v=APNkWrD-U1k Khan Academy: Proof: Law of Sines

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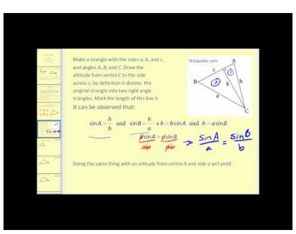
URL: http://www.ck12.org/flx/render/embeddedobject/58144

http://www.youtube.com/watch?v=dxYVBbSXYkA James Sousa: The Law of Sines: The Basics



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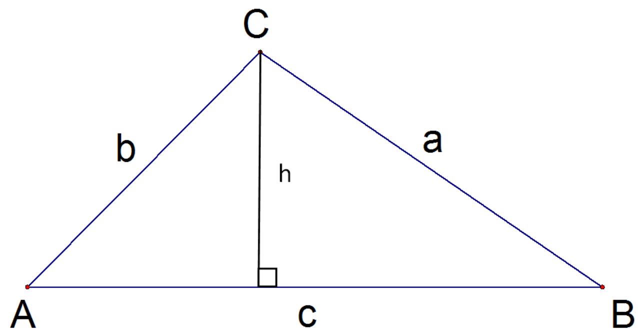
URL: http://www.ck12.org/flx/render/embeddedobject/58146

### Guidance

When given two sides and an angle that is not included between the two sides, you can use the Law of Sines. The Law of Sines states that in every triangle the ratio of each side to the sine of its corresponding angle is always the same. Essentially, it clarifies the general concept that opposite the largest angle is always the longest side.

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Here is a proof of the Law of Sines:




Looking at the right triangle formed on the left:

$$\sin A = \frac{h}{b}$$
$$h = b \sin A$$

Looking at the right triangle formed on the right:

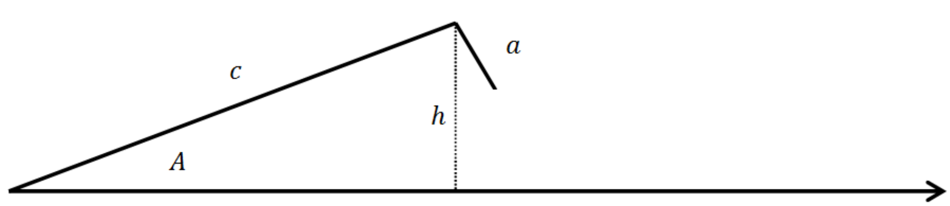
$$\sin B = \frac{h}{a}$$
$$h = a \sin B$$

Equating the heights which must be identical:

$$a \sin B = b \sin A$$
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

The best way to use the Law of Sines is to draw an extremely consistent picture each and every time even if that means redrawing and relabeling a picture. The reason why the consistency is important is because sometimes given SSA information defines zero, one or even two possible triangles.

Always draw the given angle in the bottom left with the two given sides above.




In this image side $a$ is deliberately too short, but in most problems you will not know this. You will need to compare $a$ to the height.

$$\sin A = \frac{h}{c}$$
$$h = c \ \sin A$$

**Case 1:** $a < h$

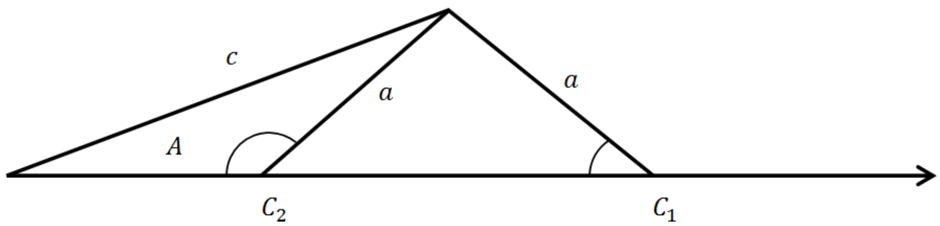
Simply put, side $a$ is not long enough to reach the opposite side and the triangle is impossible.

**Case 2:** $a = h$

Side $a$ just barely reaches the opposite side forming a $90^\circ$ angle.

**Case 3:** $h < a < c$

In this case side $a$ can swing toward the interior of the triangle or the exterior of the triangle- there are two possible triangles. This is called the ambiguous case because the given information does not uniquely identify one triangle. To solve for both triangles, use the Law of Sines to solve for angle $C_1$ first and then use the supplement to determine $C_2$.

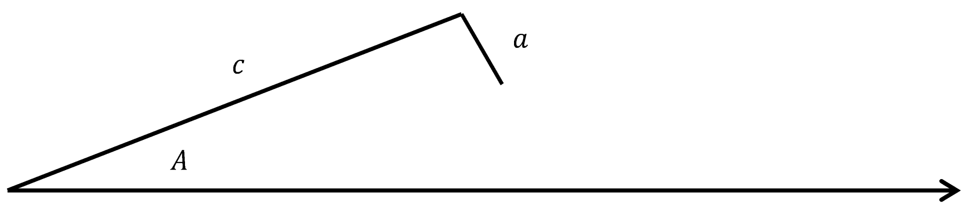



**Case 4:** $c \le a$

In this case, side $a$ can only swing towards the exterior of the triangle, only producing $C_1$.

**Example A**

$\angle A = 40^\circ, c = 13$, and $a = 2$. If possible, find $\angle C$.



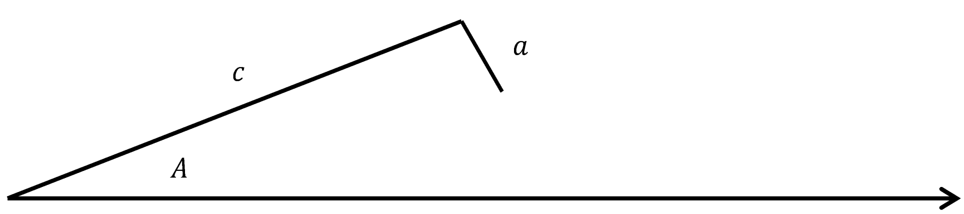

**Solution:**

$$\sin 40^\circ = \frac{h}{13}$$
$$h = 13 \sin 40^\circ \approx 8.356$$

Because $a < h$ $(2 < 8.356)$, this information does not form a proper triangle.

**Example B**

$\angle A = 17^\circ, c = 14$, and $a = 4.0932\ldots$ If possible, find $\angle C$.



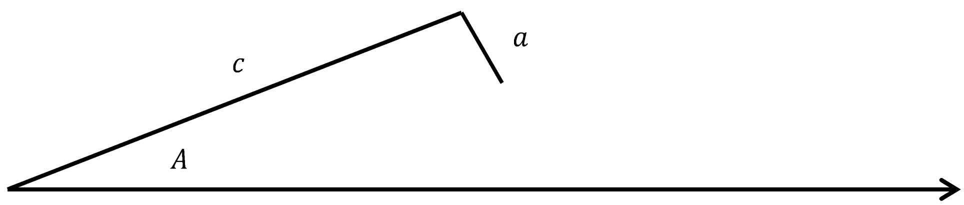
**Solution:**

$$\sin 17^\circ = \frac{h}{14}$$
$$h = 14\ \sin 17^\circ \approx 4.0932\ldots$$

Since $a = h$, this information forms exactly one triangle and angle $C$ must be $90^\circ$.

**Example C**

$\angle A = 22^\circ, c = 11$ and $a = 9$. If possible, find $\angle C$.




**Solution:**

$$\sin 22^\circ = \frac{h}{11}$$
$$h = 11\ \sin 22^\circ \approx 4.12\ldots$$

Since $h < a < c$, there must be two possible angles for angle $C$.

Apply the Law of Sines:

$$\begin{aligned}
\frac{9}{\sin 22^\circ} &= \frac{11}{\sin C_1} \\
9 \sin C_1 &= 11 \sin 22^\circ \\
\sin C_1 &= \frac{11 \sin 22^\circ}{9} \\
C_1 &= \sin^{-1}\left(\frac{11 \sin 22^\circ}{9}\right) \approx 27.24\ldots^\circ \\
C_2 &= 180 - C_1 = 152.75\ldots^\circ
\end{aligned}$$

**Concept Problem Revisited**

SSA is not a method from Geometry that shows two triangles are congruent because it does not always define a unique triangle.

## Vocabulary

***Ambiguous*** means that the given information may not uniquely identify one triangle.

### Guided Practice

1. Given $\Delta ABC$ where $A = 10^\circ, b = 10, a = 11$, find $\angle B$.

2. Given $\Delta ABC$ where $A = 12^\circ, B = 50^\circ, a = 14$ find $b$.

3. Given $\Delta ABC$ where $A = 70^\circ, b = 8, a = 3$, find $\angle B$ if possible.

**Answers:**

1. $\frac{10}{\sin B} = \frac{11}{\sin 10^\circ}$

$B = \sin^{-1}\left(\frac{10 \sin 10^\circ}{11}\right) \approx 9.08\ldots^\circ$

2. $\frac{14}{\sin 12^\circ} = \frac{b}{\sin 50^\circ}$

$b = \frac{14 \sin 50^\circ}{\sin 12^\circ} \approx 51.58\ldots$

3. $\sin 70^\circ = \frac{h}{8}$

$h = 8 \sin 70^\circ \approx 7.51\ldots$

Because $a < h$, this triangle is impossible.

### Practice

For 1-3, draw a picture of the triangle and state how many triangles could be formed with the given values.

1. $A = 30^\circ, a = 13, b = 15$

2. $A = 22^\circ, a = 21, b = 12$

3. $A = 42^\circ, a = 36, b = 37$

For 4-7, find all possible measures of $\angle B$ (if any exist) for each of the following triangle values.

4. $A = 86^\circ, a = 15, b = 11$

5. $A = 30^\circ, a = 24, b = 43$

6. $A = 48^\circ, a = 34, b = 39$

7. $A = 80^\circ, a = 22, b = 20$

For 8-12, find the length of $b$ for each of the following triangle values.

8. $A = 94^\circ, a = 31, B = 34^\circ$

9. $A = 112^\circ, a = 12, B = 15^\circ$

10. $A = 78^\circ, a = 20, B = 16^\circ$

11. $A = 54^\circ, a = 15, B = 112^\circ$

12. $A = 39^\circ, a = 9, B = 98^\circ$

13. In $\Delta ABC, b = 10$ and $\angle A = 39^\circ$. What's a possible value for $a$ that would produce two triangles?

14. In $\Delta ABC, b = 10$ and $\angle A = 39^\circ$. What's a possible value for $a$ that would produce no triangles?

15. In $\Delta ABC, b = 10$ and $\angle A = 39^\circ$. What's a possible value for $a$ that would produce one triangle?

## References

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# Chapter 7 Concepts of Statistics

## Chapter Outline



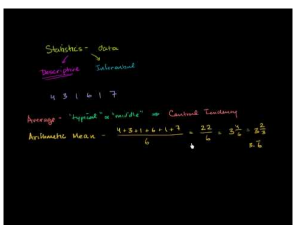
Statistics is hugely important for understanding, describing and predicting the world around you. Descriptive statistics is using summaries to present information that you have found to a reader. Summaries can be graphs or small groups of numbers that are easier to understand than long lists of numbers. Inferential statistics is using data to make predictions. Both inferential statistics and descriptive statistics help you understand the world around you and communicate it effectively.

# 7.1 Statistics on the Graphing Calculator - Finding the Mean, Median, and Mode

Here you will calculate three measures of the center of univariate data and decide which measure is best based on context. **This chapter will cover statistics and the use of the STAT mode of a graphing calculator. Please view this video which details how to enter data into a graphing calculator.** **Statistics on the Graphing Calculator** http://www.educreations.com/lesson/view/statistics-on-the-ti-84-plus-silver/15326242/?ref=appemail **Oftentimes, you are going to want to organize data in either ascending or descending order. Please view this video which covers those aspects of the STAT mode of a graphing calculator.** **Sort and Frequency Distribution Features on the Calculator** http://www.educreations.com/lesson/view/sort-frequency-distribution-feature-of-the-calcula/15334835/?ref=appemail

The three measures of central tendency are mean, median, and mode. When would it make sense to use one of these measures and not the others?

### Watch This



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/62533

http://www.youtube.com/watch?v=h8EYEJ32oQ8 Khan Academy: Statistics Intro: Mean, Median, and Mode

### Guidance

With **descriptive statistics**, your goal is to describe the data that you find in a sample or is given in a problem. Because it would not make sense to present your findings as long lists of numbers, you summarize important aspects of the data. One important aspect of the data is the **measure of central tendency**, which is a measure of the "middle" value of a set of data. There are three ways to measure central tendency:

1. Use the **mean**, which is the arithmetic average of the data.
2. Use the **median**, which is the number exactly in the middle of the data. When the data has an odd number of counts, the median is the middle number after the data has been ordered. When the data has an even number of counts, the median is the arithmetic average of the two most central numbers.
3. Use the **mode**, which is the most often occurring number in the data. If there are two or more numbers that occur equally frequently, then the data is said to be bimodal or multimodal.

Calculating the mean, median and mode is straightforward. What is challenging is determining when to use each measure and knowing how to interpret the data using the relationships between the three measures.

**Example A**

Five people were called on a phone survey to respond to some political opinion questions. Two people were from the zip code 94061, one person was from the zip code 94305 and two people were from 94062.

Which measure of central tendency makes the most sense to use if you want to state where the average person was from?

**Solution:** None of the measures of central tendency make sense to apply to this situation. Zip codes are categorical data rather than quantitative data even though they happen to be numbers. Other examples of categorical data are hair color or gender. You could argue that mode is applicable in a broad sense, but in general remember that mean, median, and mode can only be applied to quantitative data.

**Example B**

Compute the mean, median and mode for the following numbers.

3, 5, 1, 6, 8, 4, 5, 2, 7, 8, 4, 2, 1, 3, 4, 6, 7, 9, 4, 3, 2

**Solution:**

**Mean:** The sum of all these numbers is 94 and there are 21 numbers total so the mean is $\frac{94}{21} \approx 4.4762$.

**Median:** When you order the numbers from least to greatest you get:

1, 1, 2, 2, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9

The $11^{th}$ number has ten numbers to the right and ten numbers to the left so it is the median. The median is the number 4.

**Mode:** the most frequently occurring number is the number 4.

*Note: it is common practice to round to 4 decimals in AP Statistics.*

**Example C**

You write a computer code to produce a random number between 0 and 10 with equal probability. Unfortunately, you suspect your code doesn't work perfectly because in your first few attempts at running the code, it produces the following numbers:

1, 9, 1, 1, 9, 2, 9, 1, 9, 9, 9, 2, 2

How would you argue using mean, median, or mode that this code is probably not producing a random number between 0 and 10 with equal probability?

**Solution:** This question is very similar to questions you will see when you study statistical inference.

First you would note that the mean of the data is 4.9231. If the data was truly random then the mean would probably be right around the number 5 which it is. This is not strong evidence to suggest that the random number generating code is broken.

Next you would note that the median of the data is 2. This should make you suspect that something is wrong. You would expect that the median is of random numbers between 0 and 10 to be somewhere around 5.

Lastly, you would note that the mode of the data is 9. By itself this is not strong data to suggest anything. Every sample will have to have at least one mode. What should make you suspicious, however, is the fact that only two other numbers were produced and were almost as frequent as the number 9. You would expect a greater variety of numbers to be produced.

**Concept Problem Revisited**

In order to decide which measure of central tendency to use, it is a good idea to calculate and interpret all three of the numbers.

For example, if someone asked you how many people can sit in the typical car, it would make more sense to use mode than to use mean. With mode, you could find out that a five person car is the most frequent car driven and determine that the answer to the question is 5. If you calculate the mean for the number of seats in all cars, you will end up with a decimal like 5.4, which makes less sense in this context.

On the other hand, if you were finding the central heights of NBA players, using mean might make a lot more sense than mode.

### Vocabulary

The ***mean*** is the arithmetic average of the data.

The ***median*** is the number in the middle of a data set. When the data has an odd number of counts, the median is the middle number after the data has been ordered. When the data has an even number of counts, the median is the average of the two most central numbers.

The ***mode*** is the most often occurring number in the data. If there are two or more numbers which occur equally frequently, then the data is said to be ***bimodal*** or ***multimodal***.

With ***descriptive statistics***, your goal is to describe the data that you find in a sample or is given in a problem.

With ***inference statistics***, your goal is use the data in a sample to draw conclusions about a larger population.

### Guided Practice

1. Ross is with his friends and they want to play basketball. They decide to choose teams based on the number of cousins everyone has. One team will be the team with fewer cousins and the other team will be the team with more cousins. Should they use the mean, median or mode to compute the cutoff number that will separate the two teams?

2. Compute the mean, median, and mode for the following numbers.

1, 4, 5, 7, 6, 8, 0, 3, 2, 2, 3, 4, 6, 5, 7, 8, 9, 0, 6, 5, 3, 1, 2, 4, 5, 6, 7, 8, 8, 8, 4, 3, 2

3. The cost of fresh blueberries at different times of the year are:

$2.50, $2.99, $3.20, $3.99, $4.99

If you bought blueberries regularly what would you typically pay?

**Answers:**

1. Ross and his friends should use the median number of cousins as the cutoff number because this will allow each team to have the same number of players. If there are an odd number of people playing, then the extra person will just join either team or switch in later.

2. The mean is 4.6061. The median is 5. The mode is 8.

3. The word "typically" is used instead of average to allow you to make your own choice as to whether mean, median, or mode would make the most sense. In this case, mean does make the most sense. The average cost is $3.53.

### Practice

You surveyed the students in your English class to find out how many siblings each student had. Here are your results:

0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 10, 12

1. Find the mean, median, and mode of this data.

2. Why does it make sense that the mean number of siblings is greater than the median number of siblings?

3. Which measure of central tendency do you think is best for describing the typical number of siblings?

4. So far in math you have taken 10 quizzes this semester. The mean of the scores is 88.5. What is the sum of the

scores?

5. Find $x$ if 5, 9, 11, 12, 13, 14, 16, and $x$ have a mean of 12.

6. Meera drove an average of 22 miles a day last week. How many miles did she drive last week?

7. Find $x$ if 2, 6, 9, 8, 4, 5, 8, 1, 4, and $x$ have a median of 5.

Calculate the mean, median, and mode for each set of numbers:

8. 11, 15, 19, 12, 21, 34, 15, 28, 24, 15, 27, 19, 20, 13, 15

9. 3, 5, 7, 5, 5, 17, 8, 9, 11, 5, 3, 7

10. -3, 0, 5, 8, 12, 4, 2, 1, 6

Calculate the mean and median for each set of numbers:

11. 12, 88, 89, 90

12. 16, 17, 19, 20, 20, 98

13. For which of the previous two questions was the median **less than** the mean? What in the set of numbers caused this?

14. For which of the previous two questions was the median **greater than** the mean? What in the set of numbers caused this?

15. In each of the sets of numbers for problems 11 and 12, there is one number that could be considered an **outlier**. Which numbers do you think are the outliers and why? What would happen to the mean and median if you removed the outliers?

# 7.2 Standard Deviation and Mean Absolute Deviation

Here you will apply what you know about mean and averages to weighted averages and expected value.

## Standard Deviation

The Standard Deviation is a measure of how spread out numbers are.

Its symbol is $\sigma$ (the greek letter sigma)

A standard deviation close to 0 indicates that the data points tend to be very close to the mean (also called the expected value) of the set, while a high standard deviation indicates that the data points are spread out over a wider range of values.

View this video to learn more about standard deviation.

**Standard Deviation**

http://www.educreations.com/lesson/view/standard-deviation/15376333/?ref=appemail Mean Absolute Deviation

1. The **mean absolute deviation** of a set of data is the average distance between each data value and the **mean**.
2. 1. Find the **mean**.
3. 2. Find the distance between each data value and the **mean**.

View this video to learn more about the Mean Absolute Deviation (M.A.D.) **Mean absolute deviation** http://youtu.be/PwsXncM2pas

# 7.3 Five Number Summary, Box and Whisker and Modified Box Plots and Outliers

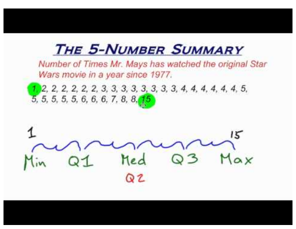
Here you will calculate quartiles and produce five number summaries for data sets.

When given a long list of numbers, it is useful to summarize the data. One way to summarize the data is to give the lowest number, the highest number and the middle number. In addition to these three numbers it is also useful to give the median of the lower half of the data and the median of the upper half of the data. These five numbers give a very concise summary of the data.

What is the five number summary of the following data?

0, 0, 1, 2, 63, 61, 27, 13

### Watch This




MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/62539

http://www.youtube.com/watch?v=XDS5TgZ4CJA The 5 Number Summary

### Guidance

Suppose you have ordered data with $m$ observations. The rank of each observation is shown by its index.

$y_1 \leq y_2 \leq y_3 \leq \cdots \leq y_m$

In data sets that are large enough, you can divide the numbers into four parts called quartiles. The quartiles of interest are the first quartile, $Q1$, the second quartile, $Q2$, and the third quartile $Q3$. The second quartile, $Q2$, is defined to be the median of the data. The first quartile, $Q1$, is defined to be the median of the lower half of the data. The third quartile, $Q3$, is similarly defined to be the median of the upper half of the data.

These three numbers in addition to the minimum and maximum values are the five number summary. Note that there are variations of the five number summary that you can study in a statistics course.

**Example A**

Compute the five number summary for the following data.

2, 7, 17, 19, 25, 26, 26, 32

**Solution:** There are 8 observations total.

- Lowest value (minimum) : 2
- $Q1 : \frac{7+17}{2} = 12$ *(Note that this is the median of the first half of the data - 2, 7, 17, 19)*
- $Q2 : \frac{19+25}{2} = 22$ *(Note that this is the median of the full set of data)*
- $Q3 : 26$ *(Note that this is the median of the second half of the data - 25, 26, 26, 32)*
- Upper value (maximum) : 32

**Example B**

Compute the five number summary for the following data:

4, 8, 11, 11, 12, 14, 16, 20, 21, 25

**Solution:** There are 10 observations total.

- Lowest value (minimum) : 4
- $Q1 : 11$ *(Note that this is the median of the first half of the data - 4, 8, 11, 11, 12)*
- $Q2 : \frac{12+14}{2} = 13$ *(Note that this is the median of the full set of data)*
- $Q3 : 20$ *(Note that this is the median of the second half of the data - 14, 16, 20, 21, 25)*
- Upper value (maximum) : 25

**Example C**

Compute the five number summary for the following data:

3, 7, 10, 14, 19, 19, 23, 27, 29

**Solution:** There are 9 observations total. To calculate $Q1$ and $Q3$, you should include the median in both the lower half and upper half calculations.

- Lowest value (minimum) : 3
- $Q1 : 10$ *(this is the median of 3, 7, 10, 14, 19)*
- $Q2 : 19$
- $Q3 : 23$ *(this is the median of 19, 19, 23, 27, 29)*
- Upper value (maximum) : 29

**Concept Problem Revisited**

To compute the five number summary, it helps to order the data.

0, 0, 1, 2, 13, 27, 61, 63

- Since there are 8 observations, the median is the average of the $4^{th}$ and $5^{th}$ observations: $\frac{2+13}{2} = 7.5$
- The lowest observation is 0.
- The highest observation is 63.
- The middle of the lower half is $\frac{0+1}{2} = 0.5$
- The middle of the upper half is $\frac{27+61}{2} = 44$

The five number summary is 0, 0.5, 7.5, 44, 63

### Vocabulary

The ***rank*** of an observation is the number of observations that are less than or equal to the value of that observation.

Data is divided into four parts by the **first quartile** $(Q1)$**, second quartile** $(Q2)$ and **third quartile** $(Q3)$**.** The **second quartile** is also known as the median.

### Guided Practice

1. Create a set of data that meets the following five number summary:

{2, 5, 9, 18, 20}

2. Compute the five number summary for the following data:

1, 1, 1, 2, 2, 3, 3, 3, 4, 4, 5, 5, 5, 6, 6, 7, 8, 9, 10, 15

3. Compute the five number summary for the following data:

1, 4, 96, 356, 2557, 9881, 14420, 20100

**Answers:**

1. Suppose there are 8 data points. The lowest point must be 2 and the highest point must be 20. The middle two points must average to be 9 so they could be 8 and 10. The second and third points must average to be 5 so they could be 4 and 6. The sixth and seventh points need to average to be 18 so they could be 18 and 18. Here is one possible answer:

2, 4, 6, 8, 10, 18, 18, 20

2. There are 20 observations.

- Lower : 1
- $Q1 : \frac{2+3}{2} = 2.5$
- $Q2 : \frac{4+5}{2} = 4.5$
- $Q3 : \frac{6+7}{2} = 6.5$
- Upper : 15

3. There are 8 observations.

- Lower : 1
- $Q1 : \frac{4+96}{2} = 50$
- $Q2 : \frac{356+2557}{2} = 1456.5$
- $Q3 : \frac{9881+14420}{2} = 12150.5$
- Upper: 20100

## Practice

Compute the five number summary for each of the following sets of data:

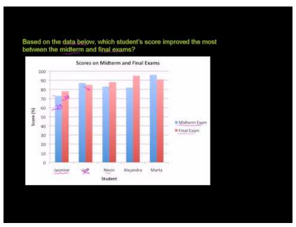
1. 0.16, 0.08, 0.27, 0.20, 0.22, 0.32, 0.25, 0.18, 0.28, 0.27
2. 77, 79, 80, 86, 87, 87, 94, 99
3. 79, 53, 82, 91, 87, 98, 80, 93
4. 91, 85, 76, 86, 96, 51, 68, 92, 85, 72, 66, 88, 93, 82, 84
5. 335, 233, 185, 392, 235, 518, 281, 208, 318
6. 38, 33, 41, 37, 54, 39, 38, 71, 49, 48, 42, 38
7. 3, 7, 8, 5, 12, 14, 21, 13, 18
8. 6, 22, 11, 25, 16, 26, 28, 37, 37, 38, 33, 40, 34, 39, 23, 11, 48, 49, 8, 26, 18, 17, 27, 14
9. 9, 10, 12, 13, 10, 14, 8, 10, 12, 6, 8, 11, 12, 12, 9, 11, 10, 15, 10, 8, 8, 12, 10, 14, 10, 9, 7, 5, 11, 15, 8, 9, 17, 12, 12, 13, 7, 14, 6, 17, 11, 15, 10, 13, 9, 7, 12, 13, 10, 12
10. 49, 57, 53, 54, 49, 67, 51, 57, 56, 59, 57, 50, 49, 52, 53, 50, 58
11. 18, 20, 24, 21, 5, 23, 19, 22
12. 900, 840, 880, 880, 800, 860, 720, 720, 620, 860, 970, 950, 890, 810, 810, 820, 800, 770, 850, 740, 900, 1070, 930, 850, 950, 980, 980, 880, 960, 940, 960, 940, 880, 800, 850, 880, 760, 740, 750, 760, 890, 840, 780, 810, 760, 810, 790, 810, 820, 850
13. 13, 15, 19, 14, 26, 17, 12, 42, 18
14. 25, 33, 55, 32, 17, 19, 15, 18, 21
15. 149, 123, 126, 122, 129, 120

# 7.4 Graphic Displays of Data

Here you will display data using bar charts, histograms, pie charts and boxplots.

Two common types of graphic displays are bar charts and histograms. Both bar charts and histograms use vertical or horizontal bars to represent the number of data points in each category or interval. The main difference graphically is that in a bar chart there are spaces between the bars and in a histogram there are not spaces between the bars. Why does this subtle difference exist and what does it imply about graphic displays in general?

### Watch This



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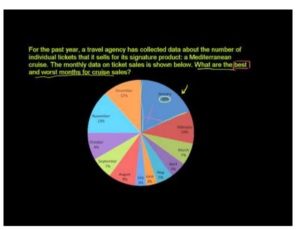
URL: http://www.ck12.org/flx/render/embeddedobject/62546

http://www.youtube.com/watch?v=kiQ6MUQZHSs Khan Academy: Reading Bar Graphs



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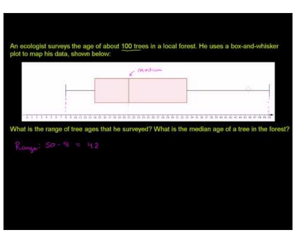
URL: http://www.ck12.org/flx/render/embeddedobject/62548

http://www.youtube.com/watch?v=4JqH55rLGKY Khan Academy: Reading Pie Graphs



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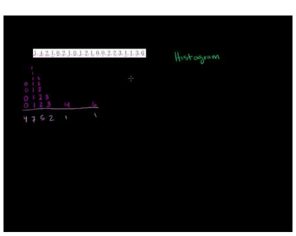
Click image to the left or use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/62550

http://www.youtube.com/watch?v=b2C9I8HuCe4 Khan Academy: Reading Box-and-Whisker Plots



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Click image to the left or use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/62552

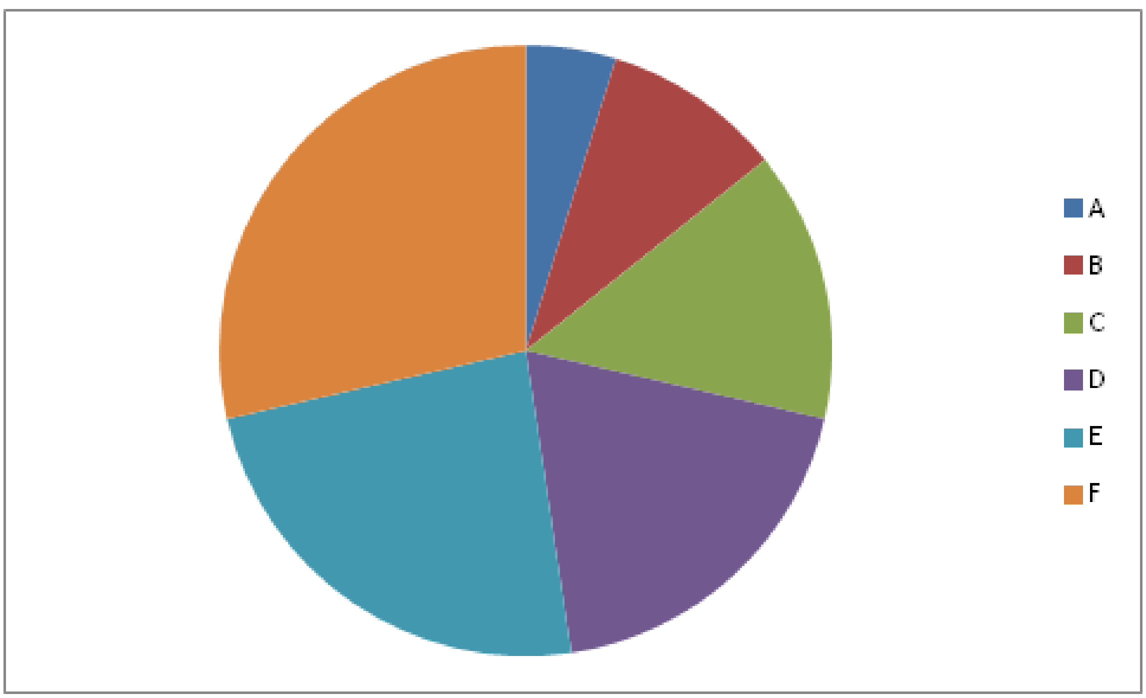
http://www.youtube.com/watch?v=4eLJGG2Ad30 Khan Academy: Histogram

### Guidance

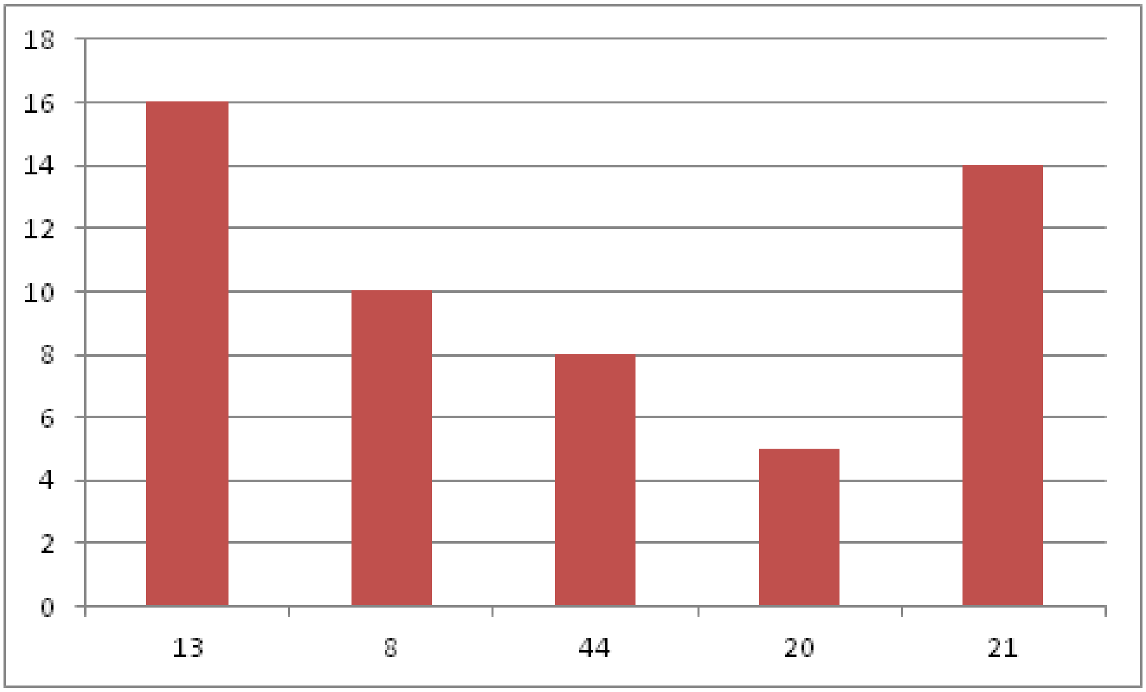
It is often easier for people to interpret relative sizes of data when that data is displayed graphically. There are a few common ways of displaying data graphically that you should be familiar with.

1) A **pie chart** shows the relative proportions of data in different categories. Pie charts are excellent ways of displaying categorical data with easily separable groups. The following pie chart shows six categories labeled $A-F$. The size of each pie slice is determined by the central angle. Since there are $360°$ in a circle, the size of the central angle $\theta_A$ of category $A$ can be found by:

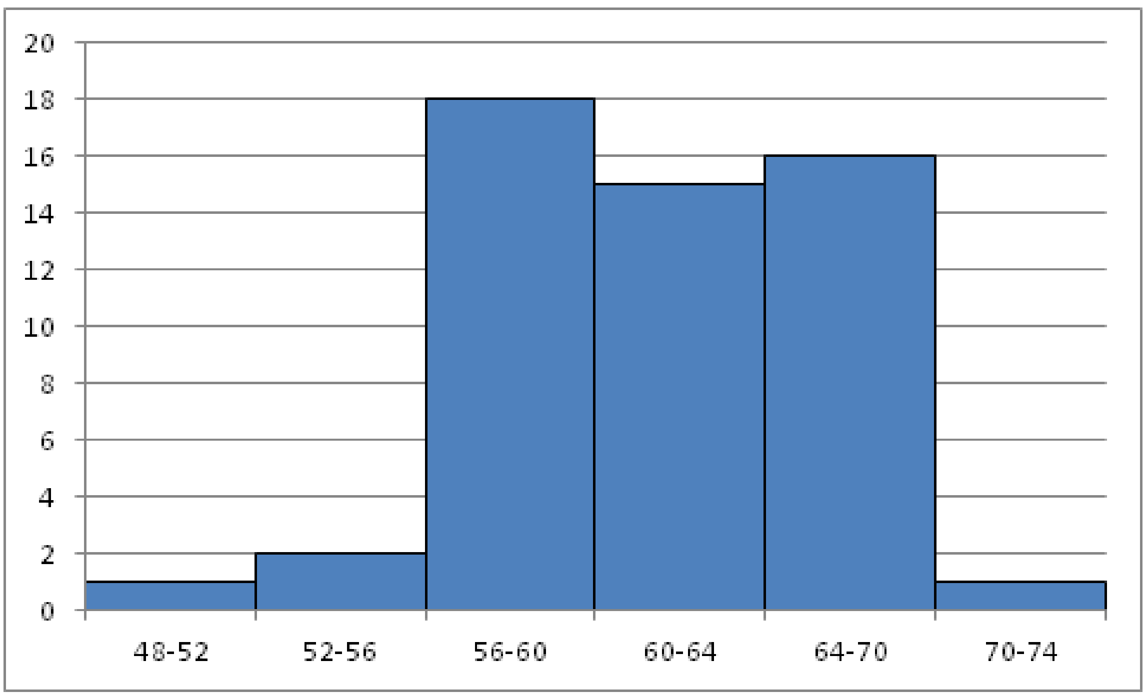
$$\frac{\theta_A}{360} = \frac{\#\ data\ points\ in\ category\ A}{Total\ number\ of\ data\ points}$$



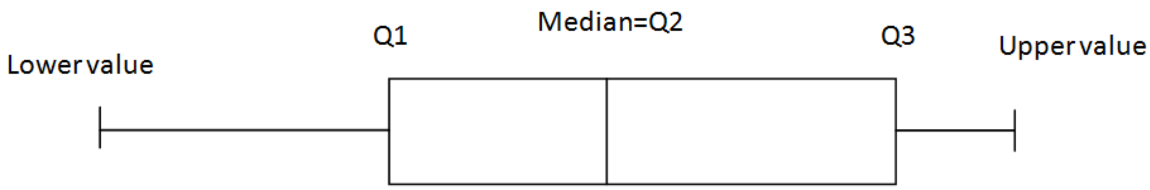

2) A **bar chart** displays frequencies of categories of data. The bar chart below has 5 categories, and shows the TV channel preferences for 53 adults. The horizontal axis could have also been labeled *News, Sports, Local News, Comedy, Action Movies.* The reason why the bars are separated by spaces is to emphasize the fact that they are categories and not continuous numbers. For example, just because you split your time between channel 8 and channel 44 does not mean on average you watch channel 26. Categories can be numbers so you need to be very careful.




3) A **histogram** displays frequencies of quantitative data that has been sorted into intervals. The following is a histogram that shows the heights of a class of 53 students. Notice the largest category is 56-60 inches with 18 people.




4) A **boxplot** (also known as a **box and whiskers plot**) is another way to display quantitative data. It displays the five 5 number summary (minimum, $Q1$, median, $Q3$, maximum). The box can either be vertically or horizontally displayed depending on the labeling of the axis. The box does not need to be perfectly symmetrical because it represents data that might not be perfectly symmetrical.

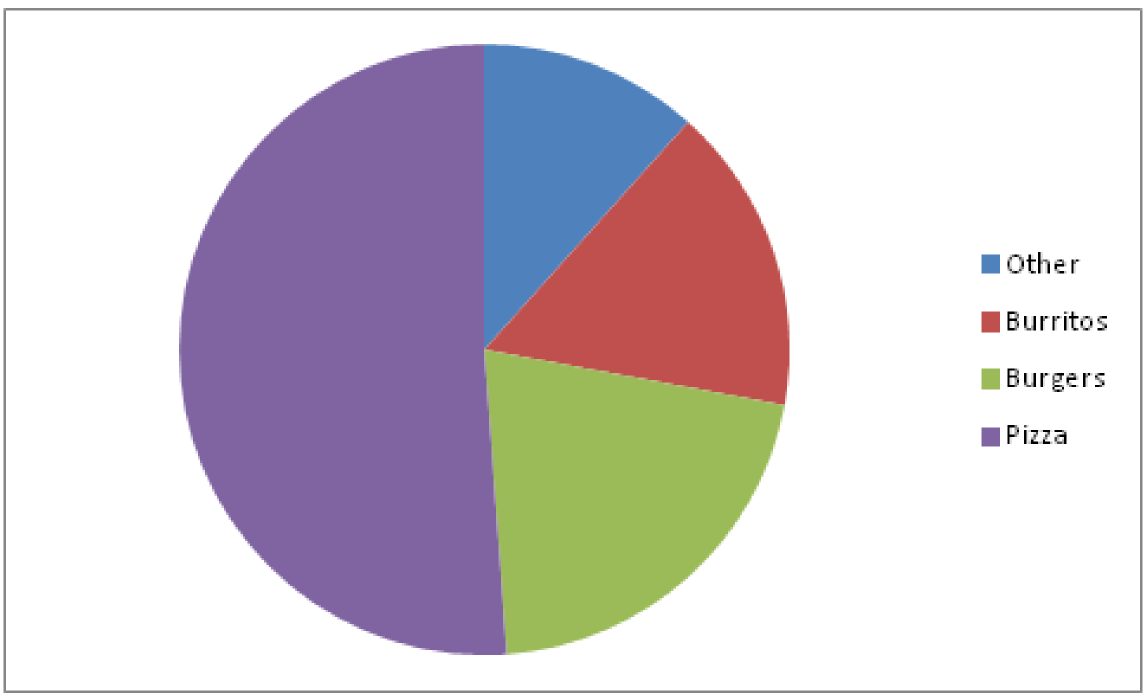



**Example A**

Create a pie chart to represent the preferences of 43 hungry students.

- Other – 5
- Burritos – 7
- Burgers – 9
- Pizza – 22

**Solution:**

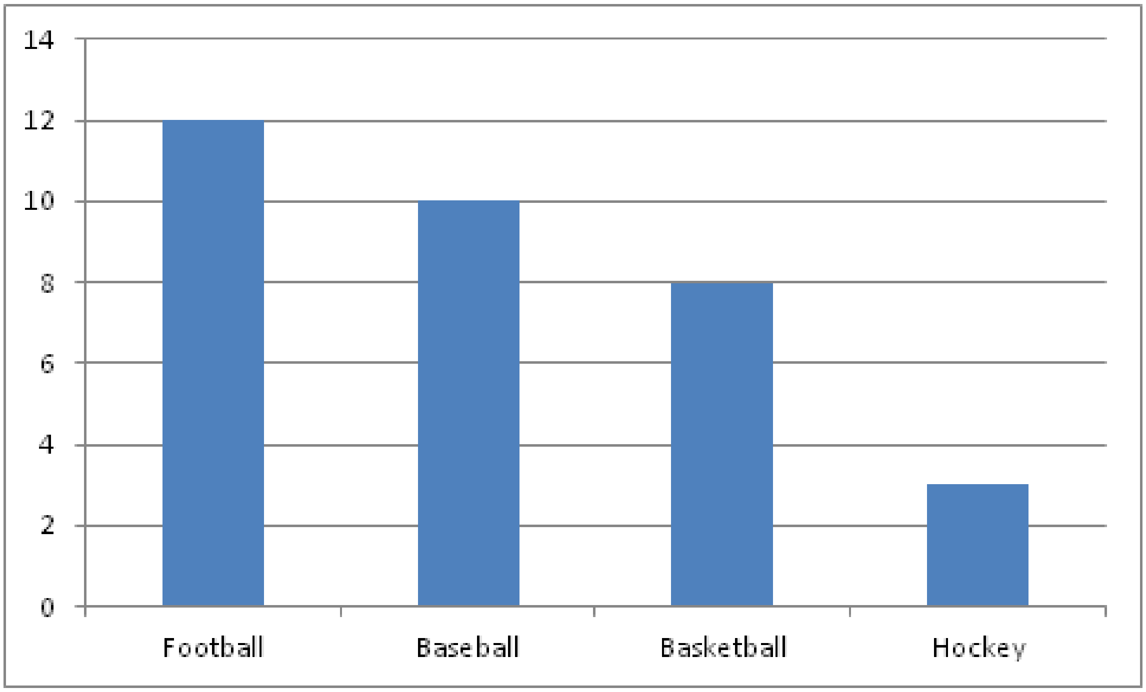



**Example B**

Create a bar chart representing the preference for sports of a group of 23 people.

- Football – 12
- Baseball – 10
- Basketball – 8
- Hockey – 3

**Solution:**

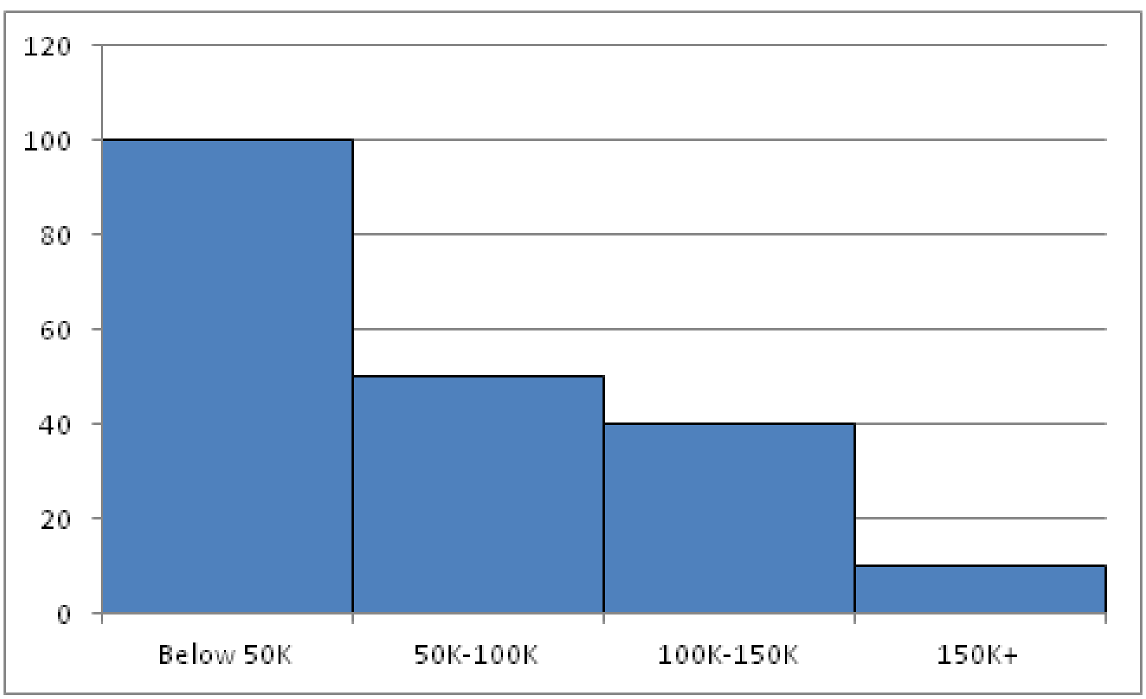



**Example C**

Create a histogram for the income distribution of 200 million people.

- Below \$50,000 is 100 million people
- Between \$50,000 and \$100,000 is 50 million people
- Between \$100,000 and \$150,000 is 40 million people
- Above \$150,000 is 10 million people

**Solution:**




**View this video to learn to create a frequency distribution on a graphing calculator. Frequency Distri-**

**butions and Histograms** http://www.educreations.com/lesson/view/frequency-distributions-histograms/15378859/?ref=appemail **Cumulative Frequency Histogram:** A **cumulative frequency** is the running total of the **frequencies**. View this video to learn more regarding cumulative frequency histograms. **Cumulative Frequency Histograms** http://www.educreations.com/lesson/view/cumulative-frequency-histograms/16865789/?ref=app

**Stem and Leaf Diagrams**

A **Stem and Leaf Plot** is a special table where each data value is split into a "stem" (the first digit or digits) and a "leaf" (usually the last digit).

**For example:**

**2.3, 2.5, 2.5, 2.7, 2.8 3.2, 3.6, 3.6, 4.5, 5.0**

And here is the stem-and-leaf plot:

TABLE 7.1:

| Stem | Leaf |
|---|---|
| 2 | 3 5 5 7 8 |
| 3 | 2 6 6 |
| 4 | 5 |
| 5 | 0 |

**View this video to learn more about Stem and Leaf Diagrams: Stem and Leaf Diagrams** http://www.educreations.com/lesson/view/stem-leaf-diagrams/15327697/?ref=appemail

**Concept Problem Revisited**

The reason for the space in bar charts but no space in histograms is bar charts graph categorical variables while histograms graph quantitative variables. It would be extremely improper to forget the space with bar charts because you would run the risk of implying a spectrum from one side of the chart to the other. Note that in the bar chart where TV stations where shown, the station numbers were not listed horizontally in order by size. This was to emphasize the fact that the stations were categories.

## Vocabulary

A ***categorical variable*** is a variable that can take on one of a limited number of values. Examples of categorical variables are tv stations, the state someone lives in, and eye color.

A ***quantitative variable*** is a variable that takes on numerical values that represent a measureable quantity. Examples of quantitative variables are the height of students or the population of a city.

A ***bar chart*** is a graphic display of categorical variables that uses bar to represent the frequency of the count in each category.

A ***histogram*** is a graphic display of quantitative variables that uses bars to represent the frequency of the count of the data in each interval.

A ***pie chart*** is a graphic display of categorical data where the relative size of each pie slice corresponds to the frequency of each category.

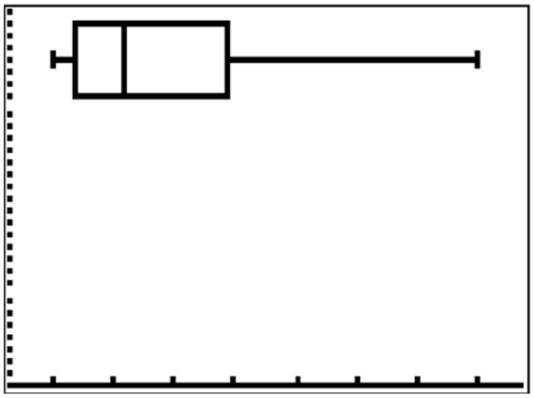
A ***boxplot*** is a graphic display of quantitative data that demonstrates the five number summary.

## Guided Practice

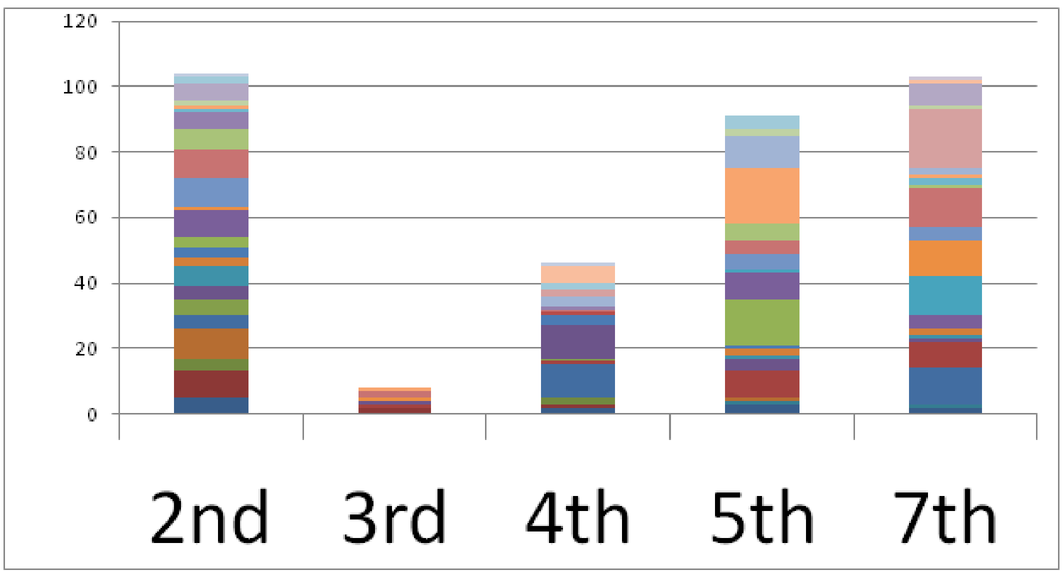
1. Create a boxplot of the following numbers in your calculator.

8.5, 10.9, 9.1, 7.5, 7.2, 6, 2.3, 5.5

2. Identify the interesting characteristics of the following boxplot.

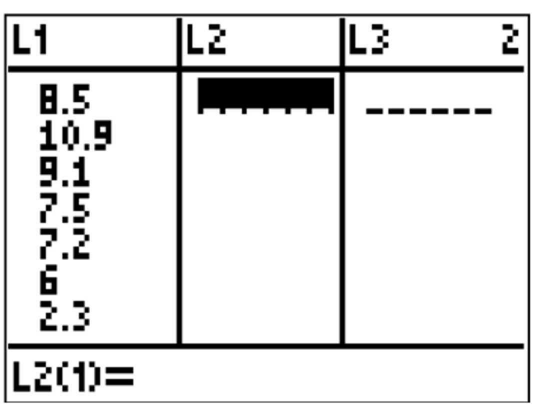


3. Interpret the following bar chart that represents the number of tardy students in 5 class periods over the course of a year.

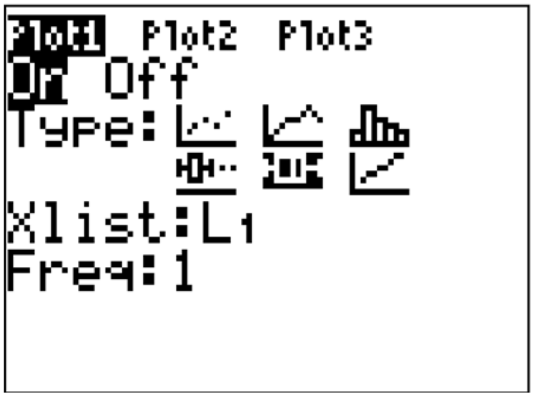



**Answers:**

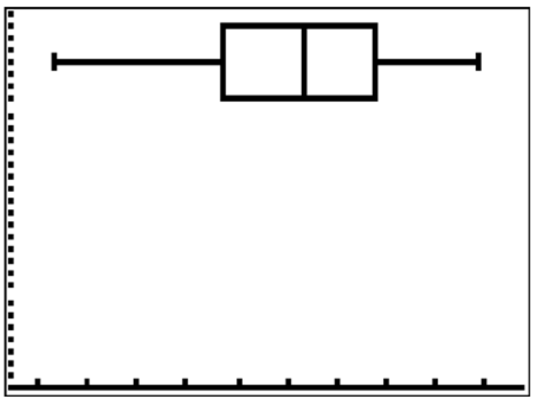
1. Enter the data into $L_1$ by going into the Stat menu.




Then turn the statplot on and choose boxplot.




Use Zoomstat to automatically center the window on the boxplot.



2. The lower bound, $Q1$ and $Q2$ all seem to be relatively close together. $Q3$ seems to be stretched a little to the right and the upper bound is significantly stretched to the right.

3. The bar chart has 5 categories representing each of the five periods. Within each category there are bands of different colors. Each band represents the number of times an individual student was tardy. For periods 5 and periods 7 there seem to be fewer students who were tardy more often. In period 2 there seems to be more students tardy a handful of times each.

### Practice

1. What types of graphs show categorical data?

2. What types of graphs show quantitative data?

A math class of 30 students had the following grades:

TABLE 7.2:

| Grade | Number of Students with Grade |
|---|---|
| A | 10 |
| B | 10 |
| C | 5 |
| D | 3 |
| F | 2 |

3. Create a bar chart for this data.

4. Create a pie chart for this data.

5. Which graph do you think makes a better visual representation of the data?

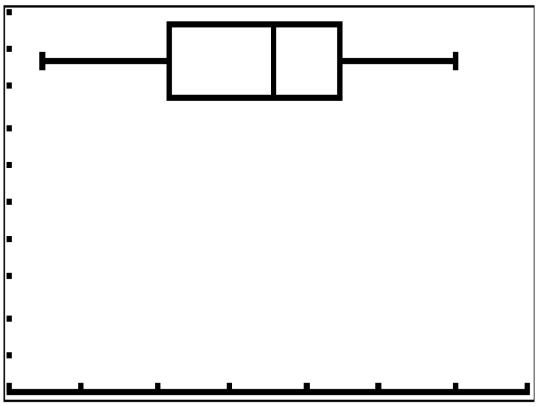
A set of 20 exam scores is 67, 94, 88, 76, 85, 93, 55, 87, 80, 81, 80, 61, 90, 84, 75, 93, 75, 68, 100, 98

6. Create a histogram for this data. Use your best judgment to decide what the intervals should be.

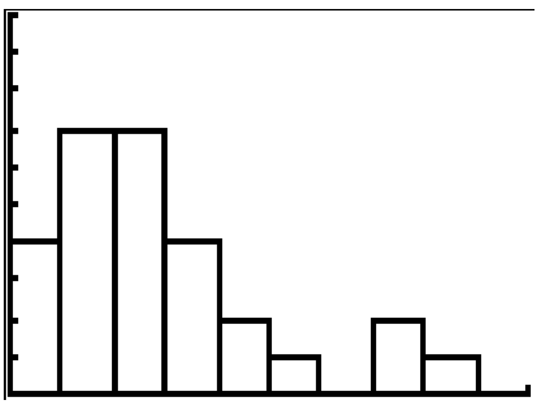
7. Find the five number summary for this data.

8. Use the five number summary to create a boxplot for this data.

9. Describe the data shown in the boxplot below.



10. Describe the data shown in the histogram below.



A math class of 30 students has the following eye colors:

### TABLE 7.3:

| Grade | Number of Students with Grade |
|---|---|
| Brown | 20 |
| Blue | 5 |
| Green | 3 |
| Other | 2 |

11. Create a bar chart for this data.

12. Create a pie chart for this data.

13. Which graph do you think makes a better visual representation of the data?

14. Suppose you have data that shows the breakdown of registered republicans by state. What types of graphs could you use to display this data?

15. From which types of graphs could you obtain information about the spread of the data? Note that spread is a

measure of how spread out all of the data is.

# 7.5 Trimmed Means and Weighted Average

Here you will calculate population variance, sample variance and standard deviation from univariate data.

Trimmed Means

A method of averaging that removes a small percentage of the largest and smallest values before calculating the mean. After removing the specified observations, the trimmed mean is found using an arithmetic averaging formula.

The trimmed mean looks to reduce the effects of outliers on the calculated average.

To Compute a 5% Trimmed Mean:

1) Order the data from smallest to largest.

2) Delete the bottom 5% of the data.

3) Delete the top 5% of the data.

4) Compute the mean of the remaining 90% of the data.

View this video for examples of calculating the 5 % Trimmed Mean and the 10 % Trimmed Mean for data: **Trimmed Means** http://www.educreations.com/lesson/view/trimmed-means/15376032/?ref=appemail

To Compute a 10% Trimmed Mean:

1) Order the data from smallest to largest.

2) Delete the bottom 10% of the data.

3) Delete the top 10% of the data.

4) Compute the mean of the remaining 80% of the data.

Weighted Average

- Not everything counts the same. For example, you final exam score "weighs more" than a regular class test does.

Definition: Weighted Average

An average in which each quantity to be averaged is assigned a weight. These weightings determine the relative importance of each quantity on the average. Weightings are the equivalent of having that many like items with the same value involved in the average.

### 'Weighted Average'

To demonstrate, let's take the value of letter tiles in the popular game Scrabble.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Value: | 10 | 8 | 5 | 4 | 3 | 2 | 1 | 0 |
| Occurrences: | 2 | 2 | 1 | 10 | 8 | 7 | 68 | 2 |

To average these values, do a weighted average using the number of occurrences of each value as the weight. To calculate a weighted average:

1. Multiply each value by its weight. *(Ans: 20, 16, 5, 40, 24, 14, 68, and 0)* 2. Add up the products of value times weight to get the total value. *(Ans: Sum=187)* 3. Add the weight themselves to get the total weight. (*Ans: Sum=100*) 4. Divide the total value by the total weight. (Ans: 187/100 = *1.87 = average value of a Scrabble tile)*

To learn more regarding weighted average, view the following video **Weighted Average** http://www.educreations.com/lesson/view/weighted-average/15375811/?ref=appemail

# 7.6 The Percentile Rank

Here you will define the standard normal distribution and learn how standard deviation and the area under the curve are connected.

View this video to receive an introduction to the concept of Percentile Rank: **Percentile rank** http://www.educreations.com/lesson/view/percentile-rank/16342497/?ref=app

Percentiles: The quartiles and the median are special cases of **percentiles** for a data set. In general, the $k$th *percentile* is a number that has $k\%$ of the data values at or below it and $(100 - k)\%$ of the data values at or above it. The lower quartile, median, and upper quartile are also the $25^{th}$ percentile, $50^{th}$ percentile, and $75^{th}$ percentile, respectively. If you are told that you scored at the $90^{th}$ percentile on a standardized test (like the SAT), it indicates that 90% of the scores were at or below your score, while 10% were at or above your score.

Use the following data values:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 284 | 586 | 987 | 412 | 256 | 541 | 312 | 251 | 444 | 695 |
| 421 | 789 | 234 | 190 | 387 | 561 | 631 | 821 | 520 | 813 |

Which number closest represents the $42^{nd}$ percentile?

**To find the value for a percentile ranking**

1) Sort the data in ascending order

2) Calculate: $\frac{\text{percentile ranking}}{100}$ X the number of data values

3) Round up to the nearest integer

4) That value calculated in step 3 is the "place" in which the answer lies.

Ex) 42/100 X 20 = 8.4 à 9

The data value in the $9^{th}$ spot is the answer. ANSWER = 421

# 7.7 Regression Analysis

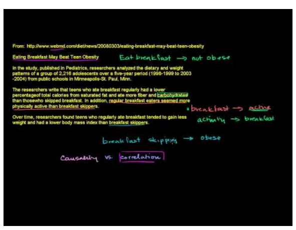
Here you will begin to work with bivariate data as you learn about linear correlation, correlation coefficients, and regression.

Statistics is largely concerned with how a change in one variable relates to changes in a second variable. Bivariate data is two lists of data that are paired up. Is there any relationship between the following data? If there is, does it mean that doctors cause cancer?

TABLE 7.4:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Number of Doctors | 27 | 30 | 36 | 60 | 81 | 90 | 156 | 221 | 347 |
| Cancer Rate | 0.02 | 0.07 | 0.16 | 0.20 | 0.43 | 0.87 | 1.21 | 2.80 | 3.91 |

### Watch This



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/62997

http://www.youtube.com/watch?v=ROpbdO-gRUo Khan Academy: Correlation vs. Causality

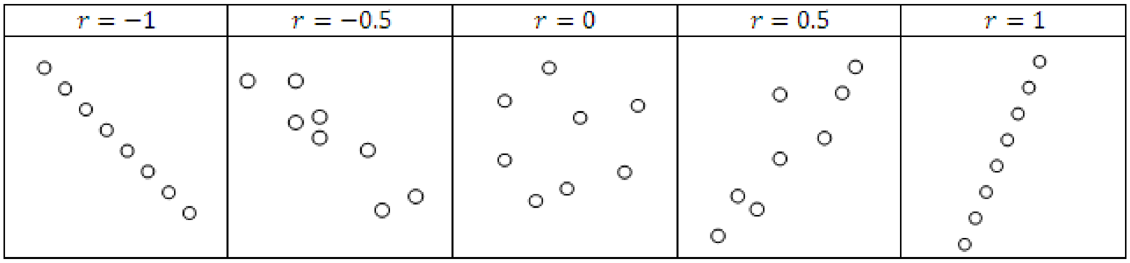
### Guidance

A **scatterplot** creates an $(x,y)$ point from each data pair. When making a scatterplot, you can try to assign the independent variable to $x$ and the dependent variable to $y$; however, it will often not be obvious which variable is the dependent variable, so you will just have to pick one.

View this video to learn how to create a scatterplot on the graphing calculator: **How to create a scatterplot on the graphing calculator** http://youtu.be/Zjj13_pKtwM **View this video to learn how to interpret a scatterplot Interpreting scatterplots** http://youtu.be/PE_BpXTyKCE Once you plot the data and zoom appropriately you will see the points scattered about. Sometimes there will be a clear linear relationship and sometimes it will appear random. The **correlation coefficient**, $r$, is a number that quantifies two aspects of the relationship between the data:

- The correlation coefficient is either negative, zero or positive. This tells you whether the data is negatively correlated, uncorrelated or positively correlated.
- The correlation coefficient is a number between $-1 \leq r \leq 1$ indicating the strength of correlation. If $r = 1$ or $r = -1$ then the data is perfectly linear. Note that a perfectly linear relationship includes lines with slopes other than 1.

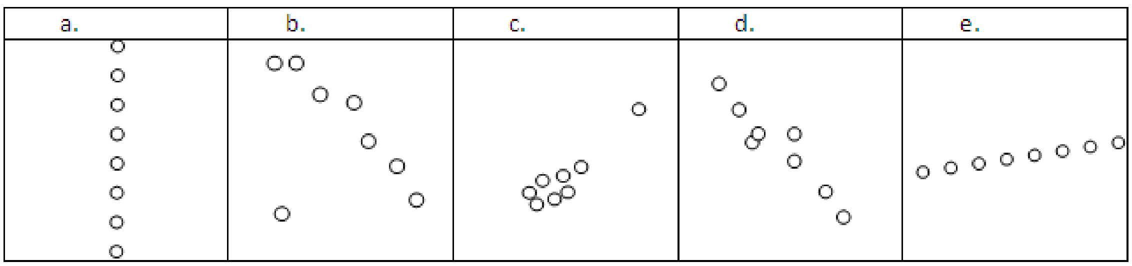
Consider the examples below to see what different correlation coefficients will look like in data:




Here are three additional videos to help you to understand correlation: **Correlation** http://www.educreations.com/lesson/view/correlation/651033/?ref=app **Types of Correlation** http://youtu.be/CWnfwZRAuaY **Examples of Correlation** http://youtu.be/AdEE07PWeEA

**Example A**

Estimate the correlation coefficient for the following scatterplots.

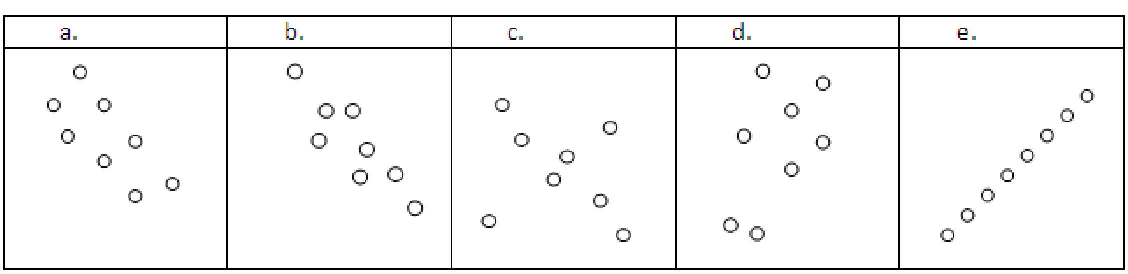



**Solution:**

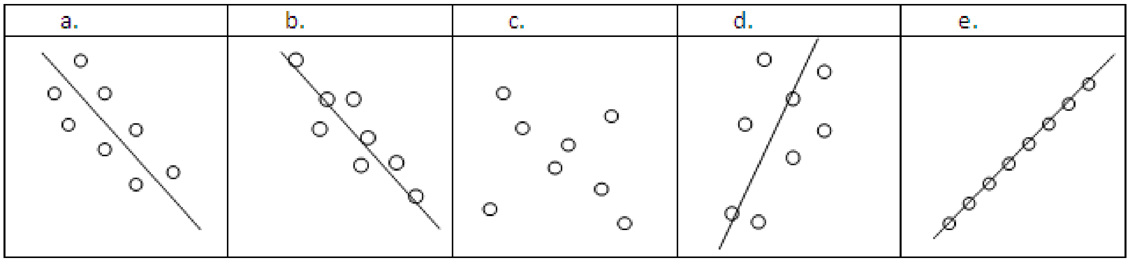
1. $r \approx 0$. Because the height $(y)$ does not seem to be dependent on the $x$, the data is uncorrelated. Another way to see this is that the slope appears to be undefined.
2. $r \approx -0.7$. If the solo point in the bottom left is an outlier, you could choose to not include it in the data. Then, the $r$ value would be closer to -1.
3. $r \approx +0.8$. The clump of data seems to be slightly positive correlated and the single point in the upper left has a strong effect indicating positive slope.
4. $r \approx -0.8$. The data seems to be fairly strongly negatively correlated.
5. $r \approx 1$. The data seems to be perfectly linearly correlated.

**Example B**

Estimate the regression line through the following scatterplots.




**Solution:** Visualize and sketch the "line of best fit" for each set of points.




Note that in part a, the regression line does not touch any point. Instead, it captures the general trend of the data. In part c, the correlation is not high enough in any direction to produce a regression line. The calculator may give a regression line for scatterplots that look like part c, but you need to be very skeptical that there is actually a relationship between the two variables.

In PreCalculus you will not learn how to calculate the correlation coefficient (you will if you take future statistics courses!). For now, the calculator will calculate it for you and your job will be to interpret the result. See Example C.

If the data is sufficiently linear, then your calculator can perform a regression to produce the equation of a line that attempts to model the trend of the data. The regression line may actually pass through all, some or none of the data points. This regression line is represented in statistics by:

y = ax + b

a represents the slope of the line

b represents the y-intercept of the line

**To do Linear Regression on the graphing calculator:**

STAT

CALC

**4: LinReg (ax b)**

**Xlist: L1 (2 1 – L1)**

**Ylist: L2 (2 2 – L2)**

**FreqList: Keep blank**

**Store RegEQ: Y1 (ALPHA F4 Y1 ENTER)**

**Calculate: Press Enter**

### Example C

Use your calculator to perform a linear regression on the following data. Then, predict the height of someone who has shoe size 9.

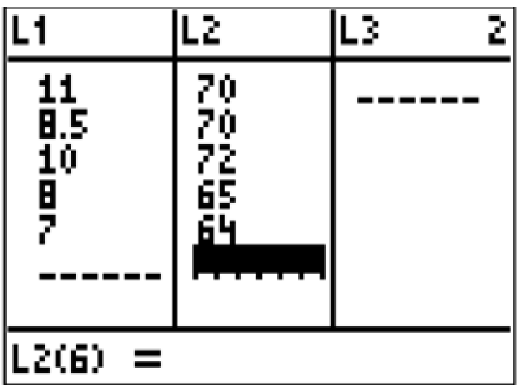
**TABLE 7.5:**

| Shoe Size | Height (in) |
|---|---|
| 11 | 70 |

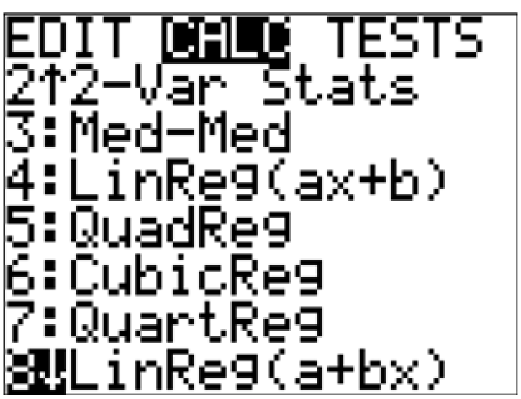
**TABLE 7.5:** (continued)

| | |
|---|---|
| 8.5 | 70 |
| 10 | 72 |
| 8 | 65 |
| 7 | 64 |

**Solution:** First enter the data.



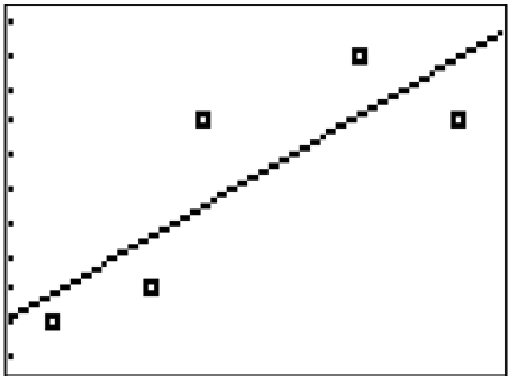

Next perform the regression. Notice that the calculator can perform linear regression in two ways that are essentially the same.




Notice that the $r$ value is about 0.8. This indicates that there is a fairly strong positive correlation between shoe size and height. *If you calculator does not display the r and $r^2$ lines then you need to go into the catalog and run the program "DiagnosticOn". This will enable the display of the correlation coefficient.*

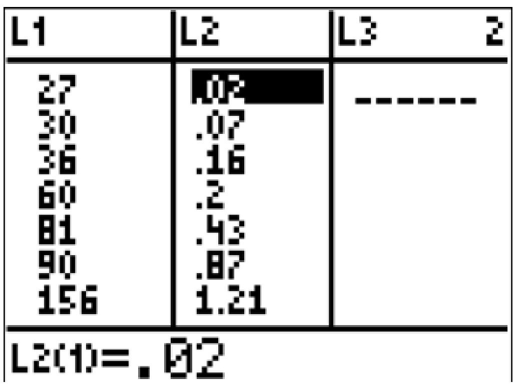
If you are not shown r and r squared on your calculator, view this video: **Getting R and R^2 on your calculator if you don't have it already** http://youtu.be/HK-gyKwNR6M

You can then graph the scatterplot and the regression line by pressing GRAPH

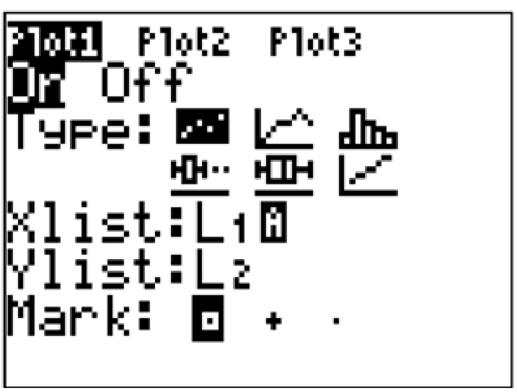


**Concept Problem Revisited**

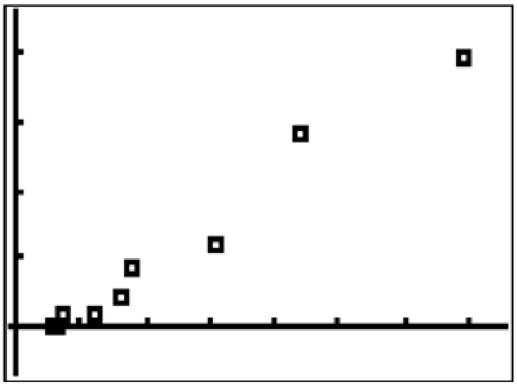
Enter the data onto lists in your calculator:




Turn the [STAT PLOT] on that compares the two lists of data:




You should note that the data is extremely linear with a positive correlation coefficient:



A naïve conclusion would be to say that doctors cause cancer. One of the most misunderstood concepts in statistics is that correlation does not imply causation. Just because there is a correlation between the number of doctors and the cancer rate doesn't mean that the number of doctors *causes* the cancer. There are dozens of reasons why more doctors might correlate with higher cancer rates. In general, remember that correlation is not the same as causation. Be careful before making any conclusions about change in one variable *causing* change in another variable.

### Vocabulary

A ***scatterplot*** creates an $(x,y)$ point from each data pair.

***Bivariate data*** is two sets of data that are paired.

The ***correlation coefficient***, $r$, is a number in the interval [-1, 1]. It indicates the strength of the correlation between two variables.

### Guided Practice

1. The data below represents the average number of working words in an elementary student's vocabulary as it relates to their shoe size. Perform a linear regression that models the data.

**TABLE 7.6:**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Shoe Size | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 | 4.5 |
| Vocabulary | 1135 | 1983 | 2501 | 4113 | 5431 | 7891 | 9320 | 11041 |

2. Use the equation from Guided Practice 1 to predict the vocabulary for someone who has a 1.0 shoe size. Does this prediction seem reasonable given the data? Why or why not?

3. Shaquille O'Neal has size 23 shoes. What, if anything can you infer about his vocabulary? Does a larger shoe size cause a larger vocabulary?

# 7.8 Quadratic and Exponential Regression

Here you will use regression on a variety of different types of data to make reasonable predictions.

For the Hyundai Elantra, a study was done to compare the speed $x$ (in mph) with the gas mileage $y$ (in miles per gallon).

| speed | mpg | speed | mpg | speed | mpg |
|---|---|---|---|---|---|
| 15 | 22.3 | 35 | 28.8 | 55 | 30.4 |
| 20 | 25.5 | 40 | 30.0 | 60 | 28.8 |
| 25 | 27.5 | 45 | 29.9 | 65 | 27.4 |
| 30 | 29.0 | 50 | 30.2 | 70 | 25.3 |

(a) Use a graphing utility to create a scatterplot.

(b) Use the regression feature to find a linear model that fits the data

(c) How well does the equation best fit the data?

Enter the speeds under L1 and the mpg under L2.

STAT EDIT

$2^{ND}$ STAT PLOT

ON

TYPE 1

Xlist: L1

Ylist: L2

Zoom

9: ZoomStat

(b) Use the regression feature to find a linear model that fits the data

STAT CALC

4:LinReg(ax+b)

Xlist: L1

Ylist: L2

FreqList: Keep Blank

Store ReqEQ: ALPHA F4 Y1

Calculate

y = ax + b

a = .0491608392

b = 25.83566434

r = .358936349

y = .05x + 25.8 (but do not round off)

(c) How well does the equation bests fit the data ?

Since r is .36, NOT very well.

If you look at the scatterplot, what type of pattern do the points seem to follow?

A QUADRATIC PATTERN

TO PERFORM A QUADRATIC REGRESSION OF DATA

STAT

CALC

5:QuadReg

Xlist: L1

Ylist: L2

FreqList: Keep Blank

Store RegEQ: ALPHA F4 Y2 (normally Y1 but

the linear eq. we did is Y1)

Calculate ENTER

$y = ax^2 + bx + c$

a = - .0081968032

b = .7458891109

c = 13.47215285

$R^2$ = .964557828 (coefficient of determination)

Two notes

1) The equation $y = -.008x^2 + .746x + 13.472$ is the quadratic equation of best fit (Don't round off – the equation is stored next to Y2 – normally Y1)

2) You must radicalize $R^2$ to get the r value.

$R^2 = .964557828$ –> $r = .9821190498$

How well does the equation bests fit the data now? VERY VERY WELL

You are going to be asked to perform regressions and are going to be asked which is the "better fit".

The regression with the greater r-value (regression coefficient) is the better fit.

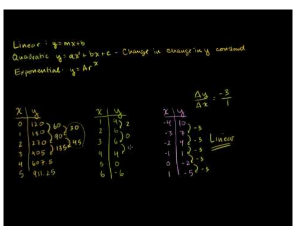
Note:

There are many different types of mathematical regressions.

Linear, Quadratic, Cubic ($x^3$), Quartic ($x^4$), logarithm, exponential, power, trigonometric, ETC.

Linear correlation is the simplest type of relationship between two variables. Your calculator has the power to use a variety of different function families to find other relationships and create many different types of models. How do you choose which function family is best for a given situation?

### Watch This



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/60152

http://www.youtube.com/watch?v=CxEFOozrMSE Khan Academy: Linear, Quadratic, and Exponential Models

### Guidance

Once you understand how to do linear regression with your calculator, you already know the technical mechanics to perform other regressions in the [STAT] [CALC] menu. The most common regressions correspond to the function families.

- QuadReg - Quadratic function family
- CubicReg - Cubic function family
- QuarticReg - Quartic function family or $4^{th}$ degree polynomial
- LnReg - Natural Log function family
- ExpReg - Exponential function family
- PwrReg - Power function family
- Logistic - Logistic function family
- SinReg - Sinusoidal function family.

When you perform these types of regressions, it will be incredibly important for you to interpret and explain parts of the graph. Here are some points to keep in mind:

1. The $y$-intercept may have a particular meaning that may or may not be reasonable.
2. When you use your model to make predictions it is important for you to remember the relevant domain of your model. If your data is about elementary school students then it might extend to middle and high school students, but it might not.
3. The calculator may produce a correlation coefficient for each of these non-linear regressions, but you should be very careful. Technically, the correlation coefficient is only supposed to be calculated with linear regression, so the calculator is doing some fancy linearization to produce it. You can learn more about this process in future statistics courses.

In general, at this point you should use your best judgment when choosing a function family to model a given set of data and deciding how good a fit the model is based on context.

**Example A**

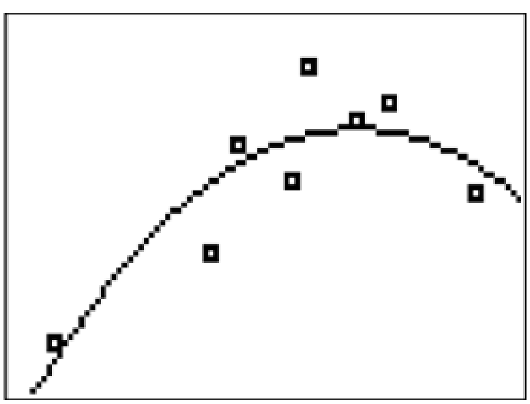
Given the following data about SAT scores and number of hours slept the night before, use an appropriate function family to produce a reasonable model. Defend your choice of function families.

### TABLE 7.7:

| # Hours Slept | SAT Score |
|---|---|
| 8.5 | 1840 |
| 10.9 | 1510 |
| 9.1 | 1900 |
| 7.5 | 2070 |
| 7.2 | 1550 |
| 6.0 | 1720 |
| 2.3 | 840 |
| 5.5 | 1230 |

**Solution:** Let $x$ be the number of hours slept and $y$ be the SAT score.

After plotting the points, you should choose a function family to use as a model. In this case, it would be appropriate to try a quadratic relationship.



A quadratic model makes sense because there seems to be a peak in the model and in the data around 8.5 hours of sleep. It makes sense that someone who does not get enough sleep will do worse and someone who gets too much sleep might also do worse.

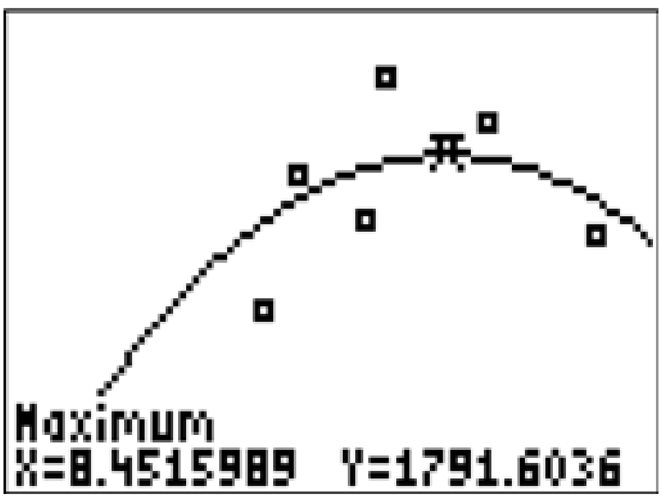
**Example B**

Using the model from Example A, answer the following questions.

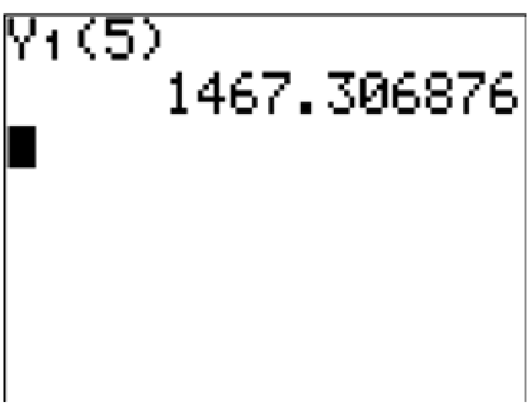
1. What is the perfect amount of sleep to get before the SATs?
2. Calculate the score you are predicted to get if you get 5 hours of sleep.
3. What is the relevant domain of the model?
4. The average SAT score is about 1500. According to the model, what amount of sleep predicts this score? Does this number represent the average number of hours that people sleep before the SATs?
5. Compare the actual and predicted score for someone with 6 hours of sleep.

**Solution:**

a. Use the calculator to find the maximum of the parabola. The $x$ coordinate represents the "perfect" amount of sleep.

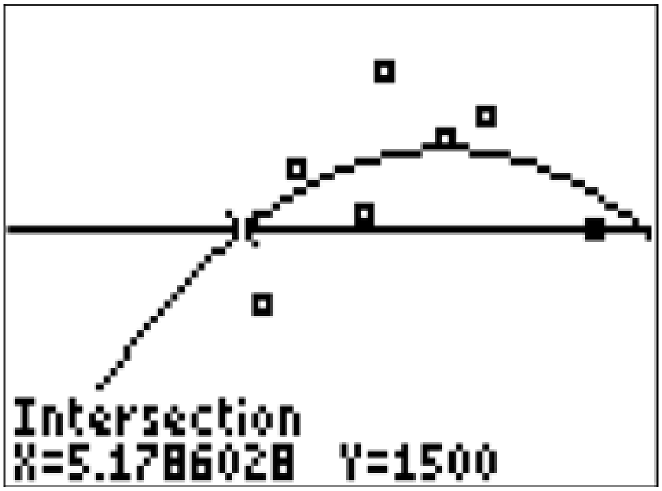



b. You can substitute $x = 5$ into the equation, or you can let the function you created and stored in $y_1$ simply act on the 5.




c. The relevant domain is between about 2 hours and 10 hours of sleep. Beyond those numbers of sleep, the model will probably not make a whole lot of sense. How could someone get negative hours of sleep?

d. You can substitute $\hat{y} = 1500$ into the equation and solve for $x$ using the quadratic formula, or you can graph the line $y = 1500$ and use the calculator to produce the two intersecting points.



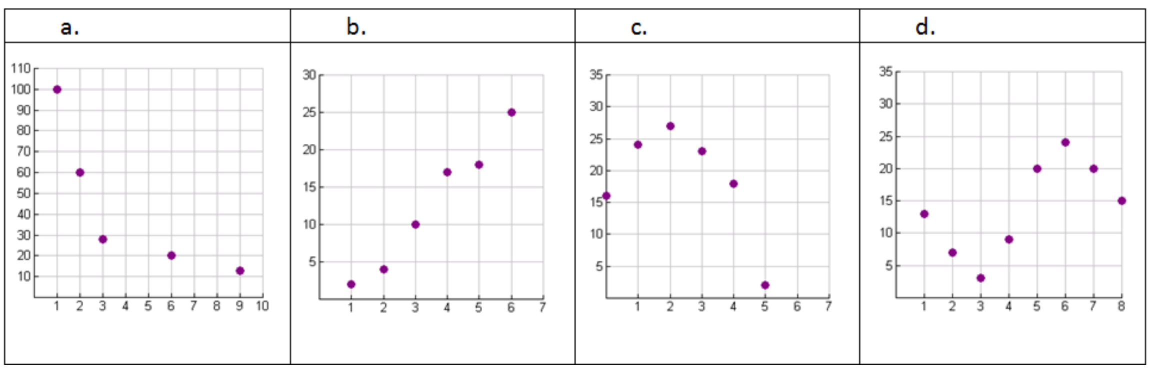

5.1786 hours and 11.7246 hours are the number hours of sleep that predict a score of 1500.

When using the model in this direction, the results do not make as much sense and you need to be extremely careful about what you say.

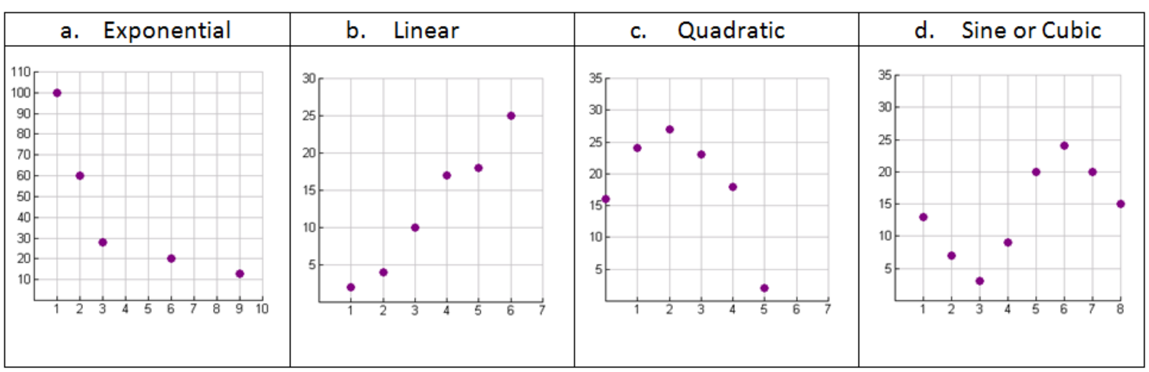
e. The actual score for someone who got 6 hours of sleep can be found in the original data to be 1720. The model predicts 1627.9970. The difference between the model and what actually happened is $1720 - 1627.9970 = 92.0030$

**Example C**

Use your knowledge of function families to predict the best model for each of the following scatterplots.

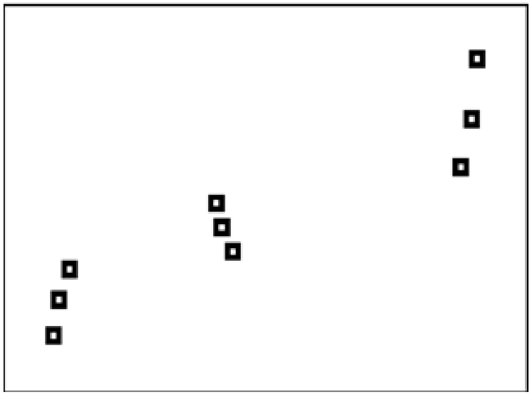


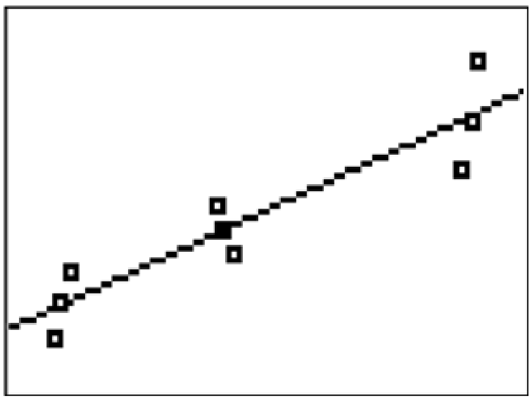

**Solution:**

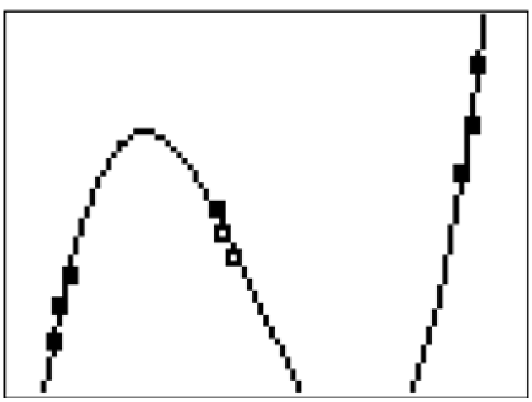



**Concept Problem Revisited**

For some data sets it is possible to use a polynomial or other complicated shape to exactly intersect every data point. The downside is that the model will miss the overall relationship. Consider the following data and modeling a linear relationship or cubic relationship.







The linear relationship describes the upward positive relationship in the data very well, but some points are slightly off of the line. The cubic relationship is much more accurate at the specific data points; however, there are features of the cubic relationship that differ significantly from reality when interpreted in context. In order to choose the best regression model you need to use context clues and the reasonableness of the various features of the model that fit each situation.

### Vocabulary

The ***residual*** is the difference between the actual height and the predicted height using the model.

### Guided Practice

1. The following data represents the height of an elephant over time. Determine the best regression function to use and determine its equation.

**TABLE 7.8:**

| Age(Years) | Height (ft) |
|---|---|
| 0 | 2 |
| 2 | 2.8 |
| 4 | 4 |
| 8 | 7.5 |
| 12 | 10 |
| 16 | 10.4 |
| 20 | 10.45 |

2. The following data represents the speed that Ben can kick a soccer ball at different ages. Determine the best regression function to use and determine its equation.

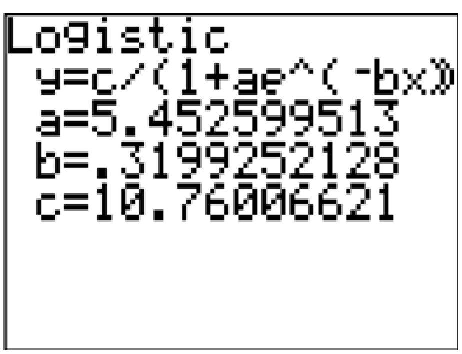
**TABLE 7.9:**

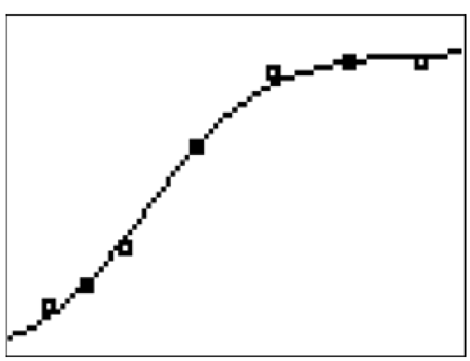
| Age(Years) | Speed (mph) |
|---|---|
| 4 | 15 |
| 10 | 32 |
| 20 | 65 |
| 30 | 70 |
| 50 | 45 |
| 60 | 35 |

3. What are two weaknesses and two strengths of the model used to predict Ben's kicking speed from Guided Practice 2?

**Answers:**

1. Logistic is the best function family because it levels off over time indicating that the elephant ceases to grow once it matures.



$$\hat{y} = \frac{10.7601}{1+5.4526 \cdot e^{-0.3199x}}$$

2. The best regression to use is a quadratic relationship because when Ben is little he cannot kick the ball very fast and when he is old he also cannot kick the ball very fast. Ben can kick the ball the fastest when he is an adult between the ages of 20 and 40.

$\hat{y} = -0.05981x^2 + 4.0679x + 0.6191$

3. One strength is that a quadratic model correctly describes the peak of kicking speed occurring in the middle of Ben's life. A linear regression might forecast Ben's kicking speed increasing forever and a logistic regression might forecast Ben's kicking speed always staying fast despite his old age. A second strength of the model could be the $y$-intercept of 0.6191. Even though this number is not really in the relevant domain, it implies that as a newborn baby Ben could kick the ball very slowly which is arguably true.

One weakness of the model is that it predicts that Ben will kick the ball at 0 miles per hour at age 68.1660. This implies that Ben will not be able to kick the ball at all which isn't necessarily true.

A second weakness of the model is that it predicts negative speed at either age extreme which doesn't make sense. A better model would be flat at 0 when Ben is born and also at the end of Ben's life when he is no longer able to kick the ball.

### Practice

The table below shows the average height of an American female by age.

**TABLE 7.10:**

| Age (Years) | Height (inches) |
|---|---|
| 2 | 34 |
| 8 | 50 |
| 11 | 57 |
| 15 | 63 |
| 23 | 64 |
| 35 | 64 |

1. Determine two different equations that model the height over time using two different function families.

2. Which function is a better fit for this data? Why?

3. Use both equations to predict the $y$-intercepts. What does the $y$-intercept represent in each case? Are your predictions reasonable for this part of the graph?

4. Use your "better fit" equation to predict the height of a 70 year-old woman. Is your prediction reasonable for this part of the graph? Why or why not? What do you really need your model to do for the domain [16,100]?

Alice is in Wonderland and drinks a potion that approximately halves her height for each sip she takes, as shown in the table below.

**TABLE 7.11:**

| # of sips | Height (inches) |
|---|---|
| 0 | 60 |
| 1 | 29 |
| 2 | 16 |
| 3 | 8 |
| 4 | 4.1 |

5. Do an exponential regression to determine an appropriate model. What is the equation?

6. Explain why exponential regression is a good choice in this case.

7. How many sips did she take if she is 2 inches tall?

8. How tall will she be if she has 6 sips?

A rumor is spreading around your 400 person school. The following table shows the number of people who know the rumor each day.

### TABLE 7.12:

| Day | # of people who know the rumor |
|---|---|
| 1 | 2 |
| 2 | 8 |
| 3 | 29 |
| 5 | 161 |
| 6 | 372 |
| 7 | 378 |
| 8 | 391 |

9. Use logistic regression to determine an equation that models the number of people who know the rumor over time.

10. Why is the logistic model appropriate in this case?

11. Use your regression equation to predict the time when only one person knew the rumor. Does this make sense?

The data table below represents how the tide changes the depth of the ocean water at a beach. At a certain place in the water, a scientist measures the depth of the water for ten consecutive hours.

### TABLE 7.13:

| Hours | Depth of Ocean Water (ft) |
|---|---|
| 0 | 9 |
| 1 | 11.2 |
| 2 | 12.4 |
| 3 | 12.9 |
| 4 | 12.5 |
| 5 | 11 |
| 6 | 8.9 |
| 7 | 7 |
| 8 | 5.5 |
| 9 | 4.9 |
| 10 | 5.4 |

12. Choose the function family that is the best model for this situation and determine the regression equation.

13. Use your regression equation to predict the depth of the water at 10 hours. What is the difference between the actual depth from the data and the predicted depth from your equation (residual)?

14. Do a cubic regression on the calculator. What is the cubic regression equation? Is this a better or worse model than the model you originally chose?

15. Why might statisticians do modeling with regression for their data?

You started by working with univariate data and learned how to display it graphically and summarize it numerically. You learned how to calculate mean, median, mode and variance, and when to use each. You also explored bivariate data and used the regression capabilities of your calculator to create mathematical models for real world phenomenon.

# CONCEPT 8 Money Problems

**There is an irrational number in math called the number "e"**

**e = 2.7281828. . . ..**

**"e" has many uses and applications such as money problems, and chemistry applications such as half-life and radioactive decay.**

**"e" has its own button on the calculator**

**2nd ln**

**2nd divided by**

**Try these - round to three decimal places**

**1)** $e^3$

**Answer: 20.086**

**2)** $-2e^{-5}$

**Answer: -0.013**

**3) 2 + e**

**Answer: 4.718**

**MONEY PROBLEMS**

**TWO INTEREST FORMULAS**

**There are two different interest formulas we will be using:**

**Interest Formula # 1**

$A = P(1 + r/n)^{nt}$

**A = Balance ($ in the account after the time has gone by)**

**P = Principal ($ invested)**

**r = Interest rate (as a decimal)**

**n = Number of compoundings per year**

**t = time in years the $ is invested for**

**Number of compoundings per year**

**quarterly n = 4**

**semiannually n = 2**

**monthly n = 12**

**weekly n = 52**

**daily n = 365**

**ex) $ 10,000 is invested at 8% for 5 years. If the interest is compounded quarterly, what will be the balance ?**

**A = P ( 1 + r/n )$^{nt}$**

**First determine your variables**

**P =**

**r =**

**n =**

**t =**

**ex) $ 10,000 is invested at 8% for 5 years. If the interest is compounded quarterly, what will be the balance ?**

**A = P ( 1 + r/n )$^{nt}$**

**P = 10,000**

**r = 0.08 (8 % as a decimal)**

**n = 4 (quarterly)**

**t = 5**

**A = 10,000 ( 1 + 0.08/4 )$^{(4)(5)}$**

**A = $ 14,859.47**

**(always round to the nearest cent)**

YOU TRY

**You deposit \$ 5,000 in a trust fund that pays 7.5 % interest for 50 years. What is the balance if the interest is compounded semiannually?**

semiannually

**A = P ( 1 + r/n )$^{nt}$**

**P = 5000**

r = .075

n = 2

t = 50

A = 5000 ( 1 + 0.075/2) $^{(2)(50)}$

**\$ 198,509.16**

**Here is a video to assist you with Interest Formula 1**

Interest Formula 1

http://www.educreations.com/lesson/view/interest-formula-1/18358604/?ref=app

**Interest Formula # 2**

**For continuous compounding of interest**

**A = Pe$^{rt}$**

**A = Balance**

**P = Principal (\$ invested)**

**r = interest rate (as decimal)**

**t = time in years**

**e = is just e (2.71828...)**

**Ex) \$ 12,000 is invested in an account which pays 7.75 % interest compounded continuously. What would the balance of the account be after 5 years ?**

**A = Pe$^{rt}$**

**P = 12,000**

**r = .0775**

**t = 5**

**$A = Pe^{rt}$**

**$A = 12{,}000e^{(.0775)(5)}$**

**ANSWER: $ 17,679.52**

**Here is a video to assist you with Interest Formula 2**

Interest formula 2

http://www.educreations.com/lesson/view/interest-formula-2/18715795/?ref=app

**Ex) Suppose you deposit**

**$ 5,000 in a trust fund that pays 7.5 % interest for 50 years. Which of the following interest options yields the greatest balance?**

**a) monthly**

**b) continuously**

a) monthly

**$A = P(1 + r/n)^{nt}$**

**P = 5000**

r = .075

n = 12

t = 50

$A = 5000(1 + 0.075/12)^{(12)(50)}$

Answer: $ 210,138.69

b) continuously

$A = Pe^{rt}$

P = 5000

R = .075

T = 50

$A = 5000e^{(.075)(50)}$

Answer: $ 212,605.41

A mortgage loan, also referred to as a mortgage, is used by purchasers of real estate when they do not possess enough money to purchase the home initially.

To calculate the monthly payment for a mortgage payment, you use the following formula:

$$\textbf{Payment} = \frac{(1 + r/12)^{12Y} \times r/12}{(1 + r/12)^{12Y} - 1}$$

The Mortgage Formula

http://www.educreations.com/lesson/view/mortgage-formula/19051558/?ref=app

In order to purchase a home, a bank requires a downpayment to be paid to the bank. In addition, included in a mortgage payment are the taxes that are owed on the home. Heres a video to explain the overall home purchase procedure:

Complete Mortgage Problems

http://www.educreations.com/lesson/view/full-mortgage-problems/19148463/?ref=app

Here is how you store the mortgage formula into the program section of the graphing calculator:

Storing the Mortgage Formula in your Programs

http://www.educreations.com/lesson/view/storing-mortgage-formula-to-the-programs-of-your-c/19148462/?ref=app